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Rotation-equivariant harmonic functions on $\mathbb{C}$

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[This document is
http://www.math.umn.edu/~garrett/m/complex/notes_2021-22/rotation-equivariant_harmonic.pdf]

[0.1] Claim: The harmonic functions u on $\mathbb{C}-\{0\}$ equivariant by $u(\mu\cdot z)=\mu^n\cdot u(z)$ for all $|\mu|=1$ are linear combinations of z^n and $\overline{z}^{-n}$, for $n\neq 0$. For $n=0$, they are linear combinations of 1 and $\log|z|$.

Proof: Grant that such u can be expressed (at least locally) as

$$u(z) \;=\; \big(z/\overline{z}\big)^{\frac{n}{2}}\cdot f(r) \qquad\qquad (\text{with } r=\sqrt{z\,\overline{z}})$$

With $\Delta=4\partial\overline{\partial}$, we embark on a slightly unpleasant but eminently feasible computation:

$$2\partial u \;=\; nz^{\frac{n}{2}-1}\overline{z}^{-\frac{n}{2}}f(r)+\big(z/\overline{z}\big)^{\frac{n}{2}}f'(r)\cdot(\overline{z}^{\frac{1}{2}}\cdot z^{-\frac{1}{2}}) \;=\; nz^{\frac{n}{2}-1}\overline{z}^{-\frac{n}{2}}f(r)+z^{\frac{n}{2}-\frac{1}{2}}\overline{z}^{-\frac{n}{2}+\frac{1}{2}}f'(r)$$

and (with some 2's and $\frac{1}{2}$'s conveniently cancelling)

$$\begin{aligned}\Delta u \;&=\; 4\overline{\partial}\partial u \;=\; 2\overline{\partial}\Big(nz^{\frac{n}{2}-1}\overline{z}^{-\frac{n}{2}}f(r)+z^{\frac{n}{2}-\frac{1}{2}}\overline{z}^{-\frac{n}{2}+\frac{1}{2}}f'(r)\Big)\\ &=\; \Big(-n^2z^{\frac{n}{2}-1}\overline{z}^{-\frac{n}{2}-1}f(r) \;+\; nz^{\frac{n}{2}-1}\overline{z}^{-\frac{n}{2}}f'(r)(\overline{z}^{-\frac{1}{2}}z^{\frac{1}{2}})\Big)\\ &\qquad +\Big((-n+1)z^{\frac{n}{2}-\frac{1}{2}}\overline{z}^{-\frac{n}{2}-\frac{1}{2}}f'(r) \;+\; z^{\frac{n}{2}-\frac{1}{2}}\overline{z}^{-\frac{n}{2}+\frac{1}{2}}f''(r)(z^{\frac{1}{2}}\cdot\overline{z}^{-\frac{1}{2}})\Big)\\ &=\; -n^2(z/\overline{z})^{n/2}\frac{1}{z\overline{z}}f(r)+(z/\overline{z})^{n/2}\frac{1}{\sqrt{z\overline{z}}}f'(r)+(z/\overline{z})^{n/2}f''(r)\end{aligned}$$

Vanishing of this expression is equivalent to

$$f''+\frac{1}{r}f'-\frac{n^2}{r^2}f \;=\; 0$$

This is an Euler-type differential equation with the indicial equation is $\lambda(\lambda-1)+\lambda-n^2=0$. For $n=0$, there is the double root 0, and corresponding linearly independent solutions $f(r)=1$ and $f(r)=\log r$. For $n\neq 0$, we have linearly independent solutions $f(r)=r^{\pm n}$.

For $n>0$, $e^{i\theta}r\to e^{in\theta}r^n$ is $z\to z^n$, and $e^{i\theta}r\to e^{in\theta}r^{-n}$ is $z\to\overline{z}^n$. For $n<0$, the roles are reversed.
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