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Review and Warm-up Exercises

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1. Prove very carefully that there is no δ-function in any (classical) sense.

2. Choose a characterization of δ and show that $x \cdot \delta = 0$. Further, show that $x \cdot \delta' = -\delta$.

3. *(Generalization of the fact that alternating decreasing sequences converge.)* Let a_n be a decreasing sequence of positive real numbers, and let b_n be a sequence of complex numbers with a constant C such that, for all N,

$$|\sum_{n \le N} b_n| \le C$$

Show that the series $\sum_n a_n b_n$ is convergent.

4. Show that $\int_0^\infty \sin(x^2)\, dx$ converges, though not absolutely.

5. Prove *(Abel's theorem)* that if $f(z)$ is a function given by a convergent power series $f(z) = \sum_{n \ge 0} c_n (z - z_o)^n$ for $|z - z_o| < R$ (with positive R), then $f'(z)$ is also given by a convergent power series in that same open disk, and it is the expected series

$$f'(z) = \sum_{n \ge 0} n c_n (z - z_o)^{n-1}$$

6. *(Natural Banach-space structure on C^k)* As usual, let $C^k[a,b]$ be the collection of k-times continuously differentiable functions on the (finite) interval $[a,b]$. Define a *norm* on $C^k[a,b]$ by

$$|f| = \sum_{i \le k} \sup_{x \in [a,b]} |f^{(i)}(x)|$$

As usual, the corresponding metric is $d(f,g) = |f - g|$. Show that with this metric $C^k[a,b]$ is a *complete* metric space. (*Hint*: you might warm up by doing $k = 0$ first.)

7. As usual, let $L^2[a,b]$ be the collection of integrable functions f on $[a,b]$ such that

$$|f|^2 = \int_a^b |f(x)|^2\, dx < \infty$$

Give an example of a sequence $\{f_n\}$ of continuous functions which is Cauchy in $L^2[a,b]$ but not Cauchy in $C^o[a,b]$.

8. Prove that $C^0[a,b]$ is dense in $L^2[a,b]$. *(Hint: Urysohn's Lemma may help.)*

9. *(Approximate identities)* Let $u_n(x)$ be a sequence of non-negative real-valued functions such that

$$\mathrm{spt}(u_{n+1}) \subset \mathrm{spt}(u_n)$$

and for any $\varepsilon > 0$ for large enough n $\mathrm{spt}(u_n) \subset [-\varepsilon, \varepsilon]$, and $\int u_n = 1$ for all n. Show that for f continuous on $\mathbf{R}$

$$\lim_n \int_{\mathbf{R}} u_n(x)\, f(x)\, dx = f(0)$$

10. *(Weak convergence)* Let

$$u_n(x) = \begin{cases} 1 & \text{for } n \le x \le n+1) \\ 0 & (\text{ else }) \end{cases}$$

Show that the sequence $\{u_n\}$ *converges weakly to* 0 in the sense that, for any $f \in \mathbf{C}_c^o(\mathbf{R})$,

$$\lim_n \int_{\mathbf{R}} u_n(x)\, f(x)\, dx = 0$$

11. Let

$$f(x) = \begin{cases} 0 & (\text{for } x \le 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

Give an attractive proof that f is infinitely differentiable at 0.