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Fourier Series

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1. Let $\{c_n : n \in \mathbf{Z}\}$ be *absolutely integrable* in the sense that

$$\sum_{n \in \mathbf{Z}} |c_n| < \infty$$

Show that

$$\sum_{n \in \mathbf{Z}} c_n \, e^{2\pi i n x}$$

converges uniformly pointwise to a continuous function.

2. Let $\{c_n : n \in \mathbf{Z}\}$ have the property that

$$\sum_{n \in \mathbf{Z}} |n \cdot c_n| < \infty$$

Show that

$$\sum_{n \in \mathbf{Z}} c_n \, e^{2\pi i n x}$$

converges uniformly pointwise to a once continuously differentiable function.

3. Show that if $\sum_n |nc_n|^2 < \infty$ then $\sum_n |c_n| < \infty$.

4. For $f \in C^1(\mathbf{R}/\mathbf{Z})$ define a Sobolev norm by

$$\|f\|_C^2 = \int_0^1 |f|^2 + \int_0^1 |f'|^2$$

Show that the completion of $C^1(\mathbf{R}/\mathbf{Z})$ with respect to this is inside $C^0(\mathbf{R}/\mathbf{Z})$. (Use the Plancherel Theorem for Fourier series.) (That is, you are proving that once-L^2-differentiability of periodic functions implies continuity.)

5. How easy might it be to find a continuous function f on $[0,1]$ with $f(0) = f(1)$ such that the Fourier coefficients c_n of f are *not* absolutely integrable?

6. First, of course, show that in a not-finite-dimensional Hilbert space the unit ball is *not* compact. The **Hilbert cube** in the Hilbert space ℓ^2 is the set

$$C = \{\{x_n\} : |x_n| \le \frac{1}{n}\}$$

Show that C is compact. Generally, for a sequence of positive real numbers ε_i show that

$$Q = \{\{x_i\} : |x_i| \le \varepsilon_i\}$$

is compact if and only if $\sum_i \varepsilon_i^2 < \infty$.

7. Recall that a **directed set** is a partially ordered set X with the property that for any two elements $x, y \in X$ there is $z \in X$ such that $x \le z$ and $y \le z$. A subset Y of X is **cofinal** if for every $x \in X$ there is $y \in Y$ such that $x \le y$. Find an X which has *no countable cofinal subset*. (*Hint:* consider the set of finite subsets of a set ordered by inclusion.)

8. A **Cauchy net** in a metric space M is a set $\{s_x : x \in X\}$ indexed by a directed set X with the property that, given $\varepsilon > 0$ there is $x_o \in X$ such that for $y, z \ge x_o$ we have $d(s_y, s_z) < \varepsilon$. Prove that if M is *complete* in the usual sense that Cauchy *sequences* converge then it is complete in the (*a priori*) stronger sense that Cauchy *nets* converge.