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Exercises on weak topologies and integrals

Paul Garrett <garrett@math.umn.edu>

1. Let V be a topological vector space and V^* its dual with the weak star topology. For $v \in V$ let B_v be the completion of V^* with respect to the seminorm

$$p_v(\lambda) = |\lambda(v)|$$

Show that B_v is one-dimensional (for $v \neq 0$). Let q_v be the natural map of V^* to B_v. Let P be the product over $v \in V$ of all these lines. Show that the map

$$\lambda \to \{q_v(\lambda) : 0 \neq v \in V\}$$

is an injection. Show that the product topology on P restricts to the weak star topology on the image of V^*.

2. Let $T : V \to W$ be a continuous linear map of topological vector spaces. Define the adjoint $T^* : W^* \to V^*$ by

$$(T^*\mu)(v) = \mu(Tv)$$

Show that T^* is continuous when both V^* and W^* are given their weak star topologies.

3. Let $G \times V \to V$ be a continuous representation of G on a topological vector space V. Define the adjoint action

$$G \times V^* \to V^*$$

on the dual space and show that it is also continuous when V^* has the weak star topology.

4. Let $V = C^o(\mathbf{R})$ be the Frechét space of continuous functions on $\mathbf{R}$, with seminorms given by suprema on compacta. (Since there is a countable cofinite set of these, the topology is Frechét.) Let $G = \mathbf{R}$ and define

$$G \times V \to V$$

by

$$\pi(g)f(x) = f(x+g)$$

Show that this map $G \times V \to V$ is continuous.

5. Let $V = C_c^o(\mathbf{R})$ be the LF space of continuous functions on $\mathbf{R}$ with compact support. Let $G = \mathbf{R}$ and define

$$G \times V \to V$$

by

$$\pi(g)f(x) = f(x+g)$$

Show that this map $G \times V \to V$ is continuous.

6. Let $V_1 \subset V_2 \subset \ldots$ be a (countable) collection of locally convex topological vector spaces such that each is closed in the next. Show that the following gives an alternative construction of the topology for the colimit V_∞ of the V_i. We construct a local basis at 0. For a collection U_i of (balanced, convex) neighborhoods of 0 in V_i (respectively), let $U \subset V_\infty$ be the convex hull of the union $\bigcup_i U_i$.