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Review of metric spaces

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

We review the basic terminology concerning metric spaces, and prove the very important *Baire category theorem*, for both *complete metric* spaces and *locally compact Hausdorff*[1] spaces.

- Metric spaces, completeness
- Completions
- Baire category theorem

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1. Metric spaces, completeness

Recall that a **metric space** X, d is a set X with a **metric** $d(,)$, a real-valued function such that, for $x, y, z \in X$,
- *(Positivity)* $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
- *(Symmetry)* $d(x, y) = d(y, x)$
- *(Triangle inequality)* $d(x, z) \leq d(x, y) + d(y, z)$ A metric space X has a *natural topology* with basis given by open balls

$$\{y \in X : d(x, y) < r\}$$

of radius $r > 0$ centered at points $x \in X$

A **Cauchy sequence** in a metric space X is a sequence $x_1, x_2, \ldots$ with the property that for every $\varepsilon > 0$ there is N sufficiently large such that for $i, j \geq N$ we have $d(x_i, x_j) < \varepsilon$. A point $x \in X$ is a **limit** of that Cauchy sequence if for every $\varepsilon > 0$ there is N sufficiently large such that for $i \geq N$ we have $d(x_i, x) < \varepsilon$. A subset X of a metric space Y is **dense** in Y if every point in Y is a limit of a Cauchy sequence in X.

The following standard lemma is often useful, and makes explicit a bit of intuition.

Lemma: Let $\{x_i\}$ be a Cauchy sequence in a metric space X, d, and suppose that the sequence converges to x in X. Given $\varepsilon > 0$, let N be sufficiently large such that for $i, j \geq N$ we have $d(x_i, x_j) < \varepsilon$. Then for $i \geq N$ we also have $d(x_i, x) \leq \varepsilon$.

Proof: Let $\delta > 0$ and take $j \geq N$ also large enough such that $d(x_j, x) < \delta$. Then for $i \geq N$ by the triangle inequality

$$d(x_i, x) \leq d(x_i, x_j) + d(x_j, x) < \varepsilon + \delta$$

Since this holds for every $\delta > 0$ we have the result. ///

A metric space is **complete** if every Cauchy sequence has a limit. [2]

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2. Completions

[1] Recall that a topological vector space is *locally compact* if every point has an open neighborhood with compact closure. A space is *Hausdorff* if for any two points x, y there are opens U, V such that $x \in U$, $y \in V$, and $U \cap V = \phi$.

[2] Convergence of Cauchy *sequences* is more properly called *sequential completeness*. In fact, for metric spaces, sequential completeness *implies* implies the strongest form of completeness, namely convergence of Cauchy *nets*, as we will observe more carefully later. This is *not* so important at the moment, but will have some importance for non-metrizable spaces, which are *rarely* complete (in the strongest sense), but in practice often are at least sequentially complete. A useful form of completeness stronger than sequential completeness but weaker than outright completeness is *local completeness*, also called *quasi-completeness*, which will play a significant role later.

A map $f : X \longrightarrow Y$ from one metric space X, d_X to another Y, d_Y is an **isometry** if it is *distance-preserving*, that is, if

$$d_Y(f(x), f(x;)) = d_X(x, x')$$

for all $x, x' \in X$. Certainly an isometry is *continuous*.

The usual definition of the **completion** Y of a metric space X is that Y is a complete metric space with an *isometry* $i : X \longrightarrow Y$ such that the image $i(X)$ is *dense*.[3]

Before describing any *construction* of a completion, we can prove some things about the behavior of any *possible* completion. In particular, we will prove that any two completions are naturally isometric to each other. Thus, whatever choice of construction we make the outcome will be the same.

Proposition: Let $i : X \longrightarrow Y$ and $j : X \longrightarrow Z$ be two completions of a metric space X. Then there is a unique bijective isometry $h : Y \longrightarrow Z$ such that

$$j = h \circ i$$

Proof: Given $y \in Y$, choose a Cauchy sequence x_k in X such that $i(x_k)$ converges to y, and *try* to define

$$h(y) = \lim_k j(x_k)$$

Even though we may anticipate that this will work fine, it is not *a priori* clear that the limit exists, that it is well-defined, etc. Although nothing surprising happens, we check those details, as follows.

Since the map j preserves distances, the sequence $j(x_k)$ is Cauchy in Z, so has a limit since Z is complete. For well-definedness, for x_k and x'_k two Cauchy sequences whose images $i(x_k)$ and $i(x'_k)$ approach y, since i is an isometry eventually x_k is close to x'_k. Thus,$j(x_k)$ is close to $j(x'_k)$ by continuity. Thus, $h(y)$ is well-defined.

To show that h is an isometry, let $y, y' \in Y$, with two Cauchy sequences x_t and x'_t approaching y and y' respectively. Given $\varepsilon > 0$, let N be large enough such that for $r, s \geq N$ we have $d_Z(h(i(x_r)), h(i(x_s))) < \varepsilon$ and $d_Z(h(i(x'_r)), h(i(x'_s))) < \varepsilon$ where $d_Z(,)$ is the metric in Z. Then (from the lemma above!) for such r also

$$d_Z(h(i(x_r)), h(y)) \leq \varepsilon$$

and

$$d_Z(h(i(x'_r)), h(y')) \leq \varepsilon$$

By the triangle inequality

$$d_Z(h(y), h(y')) \leq d_Z(h(y), h(i(x_r))) + d_Z(h(i(x_r)), h(i(x'_r))) + d_Z(h(i(x'_r)), h(y')) \leq \varepsilon + d(x_r, x'_r) + \varepsilon$$

since $j = h \circ i$ is an isometry $X \longrightarrow Z$. But also, letting $d_Y(,)$ be the metric on Y,

$$d(x_r, x'_r) = d_Y(i(x_r), i(x'_r)) \leq d_Y(i(x_r), y) + d_Y(y, y') + d_Y(i(x'_r), y')$$

and

$$d(x_r, x'_r) = d_Y(i(x_r), i(x'_r)) \geq -d_Y(i(x_r), y) + d_Y(y, y') - d_Y(i(x'_r), y')$$

so

$$|d(x_r, x'_r) - d_Y(y, y')| \leq 2\varepsilon$$

Thus

$$d_Z(h(y), h(y')) \leq d_Y(y, y') + 4\varepsilon$$

[3] The usual discussion of *completion* thus may accidentally neglect questions of uniqueness.

This proves that $h : Y \longrightarrow Z$ is an isometry. In particular, it is injective.

Now claim that $h : Y \longrightarrow Z$ is a *surjection*. Indeed, if $j(x_k)$ is a Cauchy sequence approaching a point $z \in Z$, then x_k is Cauchy in X since j is an isometry. Then $i(x_k)$ is Cauchy in Y with some limit y, and $h(y) = z$ by the definition of h. In summary, the natural definition

$$h(\lim_k i(x_k)) = \lim_k j(x_k)$$

gives a bijective isometry from the one completion to the other. ///

Now we give the standard *construction* of a completion of X. Let C be the collection of Cauchy sequences in X. Let $\sim$ be the relation on Cauchy sequences defined by $\{x_s\} \sim \{y_t\}$ if and only if for every $\varepsilon > 0$ there is N sufficiently large such that for $r, s \geq N$ we have $d(x_r, y_s) < \varepsilon$. Attempt to define a metric D on $C/\sim$ by

$$D(\{x_s\}, \{y_t\}) = \lim_s d(x_s, y_s)$$

We must verify that this is well-defined on the quotient $C/\sim$ and gives a metric. We have an injection $i : X \longrightarrow C/\sim$ by

$$x \longrightarrow \{x, x, x, \ldots\} \bmod \sim$$

We should prove that this is an isometry, and that $C/\sim$ really is *complete*.

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3. The Baire category theorem

This standard result is both indispensable and mysterious.

A set E in a topological space X is **nowhere dense** if its closure $\bar{E}$ contains no non-empty open set. A *countable union* of nowhere dense sets is said to be **of first category**, while every other subset (if any) is **of second category**. The idea (not at all clear from this traditional terminology) is that first category sets are *small*, while second category sets are *large*. In this terminology, the theorem's assertion is equivalent to the assertion that (non-empty) *complete metric* spaces and *locally compact Hausdorff* spaces are *of second category.*

Further, a G_δ set is a countable intersection of open sets. Concommitantly, an F_σ set is a countable union of closed sets. Again, the following theorem can be paraphrased as asserting that, in a complete metric space, *a countable intersection of dense G_δ's is still a dense G_δ.*

Theorem: *(Baire category)* Let X be a set with metric d making X a *complete metric space.* Or let X be a locally compact Hausdorff topological space. The intersection of a *countable* collection $U_1, U_2, \ldots$ of *dense open subsets* U_i of X is still *dense* in X.

Proof: Let B_o be a non-empty open set in X, and show that $\bigcap_i U_i$ meets B_o. Suppose that we have inductively chosen an open ball B_{n-1}. By the denseness of U_n, there is an open ball B_n whose closure $\overline{B_n}$ satisfies

$$\overline{B_n} \subset B_{n-1} \cap U_n$$

Further, for complete metric spaces, take B_n to have radius less than $1/n$ (or any other sequence of reals going to 0), and in the locally compact Hausdorff case take B_n to have compact closure.

Let

$$K = \bigcap_{n \geq 1} \overline{B_n} \subset B_o \cap \bigcap n \geq 1 U_n$$

For complete metric spaces, the centers of the nested balls B_n form a Cauchy sequence (since they are nested and the radii go to 0). By completeness, this Cauchy sequence *converges*, and the limit point lies inside each *closure* $\overline{B_n}$, so lies in the intersection. In particular, K is non-empty. For locally compact Hausdorff spaces, the intersection of a nested family of non-empty compact sets is non-empty, so K is non-empty, and B_o necessarily meets the intersection of the U_n. ///
