(*July 25, 2011*)

Topological vectorspaces

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

- Natural non-Fréchet spaces
- Topological vector spaces
- Quotients and linear maps
- More topological features
- Finite-dimensional spaces
- Seminorms and Minkowski functionals
- Countably normed, countably Hilbert spaces
- Local countability

This is the first introduction to *topological vectorspace* in generality. This would be motivated and useful *after* acquaintance with Hilbert spaces, Banach spaces, Fréchet spaces, to understand important examples *outside* these classes of spaces.

Basic concepts are introduced which make sense *without* a metric. Some concepts *appearing* to depend a metric are given sense in a general context.

Even in this generality, *finite-dimensional* topological vectorspaces have just one possible topology. This has immediate consequences for maps to and from finite-dimensional topological vectorspaces.

All this works with mild hypotheses on the scalars involved.

---

1. *Natural non-Fréchet spaces*

There are many natural spaces of *functions* that are *not* Fréchet spaces.

For example, let

$$C_c^o(\mathbb{R}) = \{\text{compactly-supported continuous } \mathbb{C}\text{-valued functions on } \mathbb{R}\}$$

This is a strictly smaller space than the space $C^o(\mathbb{R})$ of *all* continuous functions on $\mathbb{R}$, which we saw *is* Fréchet. This function space is an *ascending union*

$$C_c^o(\mathbb{R}) = \bigcup_{N=1}^{\infty} \{f \in C_c^o(\mathbb{R}) : \text{spt} f \subset [-N, N]\}$$

Each space

$$C_N^o = \{f \in C_c^o(\mathbb{R}) : \text{spt} f \subset [-N, N]\} \subset C^o[-N, N]$$

is strictly smaller than the space $C^o[-N, N]$ of *all* continuous functions on the interval $[-N, N]$, since functions in C_N^o must vanish at the endpoints. Still, C_N^o is a *closed* subspace of the Banach space $C^o[-N, N]$ (with sup norm), since a sup-norm limit of functions vanishing at $\pm N$ must also vanish there. Thus, each individual C_N^o *is* a *Banach* space.

For $0 < M < N$ the space C_M^o is a *closed* subspace of C_N^o (with sup norm), since the property of vanishing off $[-M, M]$ is preserved under sup-norm limits.

But for $0 < M < N$ the space C_M^o is *nowhere dense* in C_N^o, since an open ball of radius $\varepsilon > 0$ around any function in C_N^o contains many functions with non-zero values off $[-M, M]$.

Thus, the *set* $C_c^o(\mathbb{R})$ is an ascending union of a countable collection of subspaces, each closed in its successor, but nowhere-dense there.

Though the topology on $C_c^o(\mathbb{R})$ is not specified yet, *any* acceptable topology on $C_c^o(\mathbb{R})$ should give subspace C_M^o its natural (Banach-space) topology. *Then $C_c^o(\mathbb{R})$ is a countable union of nowhere-dense subsets.* By the Baire category theorem the topology on $C_c^o(\mathbb{R})$ cannot be complete metric. In particular, it cannot be Fréchet.

Nevertheless, the space $C_c^p(\mathbb{R})$ and many similarly-constructed spaces *do* have a reasonable structure, being an ascending union of a countable collection of Fréchet spaces, each closed in the next. [1]

[1.0.1] Remark: The space of integrals against regular Borel measures on a σ-compact [2] topological space X can be construed (either *defined* or *proven* [3] depending on one's choice) to be all continuous linear maps $C_c^o(X) \to \mathbb{C}$. This motivates understanding the topology of $C_c^o(X)$, and, thus, to understand non-Fréchet spaces.

[1.0.2] Remark: A similar argument proves that the space $C_c^\infty(\mathbb{R}^n)$ of **test functions** (compactly-supported infinitely differentiable functions) on $\mathbb{R}^n$ cannot be Fréchet. These functions play a central role in the study of *distributions* or *generalized functions*, providing further motivation to accommodate non-Fréchet spaces.

---

2. *Topological vectorspaces*

For the moment, the *scalars* need not be real or complex, need not be locally compact, and need not be commutative. Let k be a division ring. Any k-module V is a *free* k-module. [4] *We will substitute k-vectorspace for k-module in what follows.*

Let the scalars k have a **norm** $|\,|$, a non-negative real-valued function on k such that

$$\left\{ \begin{array}{l} |x| = 0 \implies x = 0 \\ \\ |xy| \;=\; |x||y| \\ \\ |x+y| \;\;\leq |x| + |y| \end{array} \right\} \qquad\qquad \text{(for all } x, y \in k)$$

Further, suppose that with regard to the metric

$$d(x,y) \;=\; |x-y|$$

the topological space k is *complete* and *non-discrete*. The non-discreteness is that, for every $\varepsilon > 0$ there is $x \in k$ such that

$$0 < |x| < \varepsilon$$

A **topological vector space** V (over k) is a k-vectorspace V with a topology on V in which *points are closed*, and so that **scalar multiplication**

$$x \times v \longrightarrow xv \qquad\qquad \text{(for } x \in k \text{ and } v \in V)$$

---

[1] A countable ascending union of Fréchet spaces, each closed in the next, suitably topologized, is an **LF-space**. This stands for *limit of Fréchet*. The topology on the union is a *colimit*, discussed a bit later.

[2] As usual, *σ-compact* means that the space is a countable union of compacts.

[3] This is the Riesz-Markov-Kakutani theorem.

[4] The proof of this free-ness is the same as the proof that a vector space over a (commutative) field is free, that is, has a basis. The argument is often called the *Lagrange replacement principle*, and succeeds for infinite-dimensional vector spaces, granting the Axiom of Choice.

and **vector addition**

$$v \times w \to v + w \qquad \text{(for } v, w \in V)$$

are *continuous.*

For subsets X, Y of V, let

$$X + Y \;=\; \{x + y : x \in X, y \in Y\}$$

Also, write

$$-X \;=\; \{-x : x \in X\}$$

The following idea is elementary, but indispensable. Given an open neighborhood U of 0 in a topological vectorspace V, continuity of vector addition yields an open neighborhood U' of 0 such that

$$U' + U' \subset U$$

Since $0 \in U'$, necessarily $U' \subset U$. This can be repeated to give, for any positive integer n, an open neighborhood U_n of 0 such that

$$\underbrace{U_n + \ldots + U_n \subset U}_{n}$$

In a similar vein, for fixed $v \in V$ the map $V \to V$ by $x \to x + v$ is a *homeomorphism*, being invertible by the obvious $x \to x - v$. Thus, *the open neighborhoods of* v *are of the form* $v + U$ *for open neighborhoods* U *of* 0. In particular, *a local basis at* 0 *gives the topology on a topological vectorspace.*

[2.0.1] Lemma: Given a compact subset K of a topological vectorspace V and a closed subset C of V not meeting K, there is an open neighborhood U of 0 in V such that

$$\text{closure}(K + U) \cap (C + U) \;=\; \emptyset$$

Proof: Since C is closed, for $x \in K$ there is a neighborhood U_x of 0 such that the neighborhood $x + U_x$ of x does not meet C. By continuity of vector addition

$$V \times V \times V \to V \quad \text{by} \quad v_1 \times v_2 \times v_3 \to v_1 + v_2 + v_3$$

there is a smaller open neighborhood N_x of 0 so that

$$N_x + N_x + N_x \subset U_x$$

By replacing N_x by $N_x \cap -N_x$, which is still an open neighborhood of 0, suppose that N_x is *symmetric* in the sense that $N_x = -N_x$.

Using this symmetry,

$$(x + N_x + N_x) \cap (C + N_x) = \emptyset$$

Since K is compact, there are finitely-many $x_1, \ldots, x_n$ such that

$$K \subset (x_1 + N_{x_1}) \cup \ldots \cup (x_n + N_{x_n})$$

Let

$$U = \bigcap_i N_{x_i}$$

Since the intersection is finite, this is open. Then

$$K + U \;\subset\; \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + U) \;\subset\; \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + N_{x_i})$$

These sets do not meet $C+U$, by construction, since $U \subset N_{x_i}$ for all i.

Finally, since $C+U$ is a union of opens $y+U$ for $y \in C$, it is open, so even the *closure* of $K+U$ does not meet $C+U$. ///

[2.0.2] Corollary: A topological vectorspace is *Hausdorff*. (Take $K = \{x\}$ and $C = \{y\}$ in the lemma). ///

[2.0.3] Corollary: The topological closure $\bar{E}$ of a subset E of a topological vectorspace V is obtained as

$$\bar{E} = \bigcap_U E + U$$

where U ranges over a local basis at 0.

Proof: In the lemma, take $K = \{x\}$ and $C = \bar{E}$ for a point x of V not in C. Then we obtain an open neighborhood U of 0 so that $x+U$ does not meet $\bar{E}+U$. The latter contains $E+U$, so certainly $x \notin E+U$. That is, for x not in the closure, there is an open U containing 0 so that $x \notin E+U$. ///

[2.0.4] Remark: It is convenient that Hausdorff-ness of topological vectorspaces follows from the weaker assumption that points are closed.

---

3. *Quotients and linear maps*

We continue to suppose that the *scalars* k are a *non-discrete complete normed division ring*. It suffices to think of $\mathbb{R}$ or $\mathbb{C}$.

For two topological vectorspaces V, W over k, a function

$$f : V \to W$$

is **(k-)linear** when

$$f(\alpha x + \beta y) \;=\; \alpha f(x) + \beta f(y)$$

for all $\alpha, \beta \in k$ and $x, y \in V$. Almost without exception we will be interested exclusively in **continuous** linear maps, meaning linear maps continuous for the topologies on V, W. The **kernel** $\ker f$ of a linear map is

$$\ker f = \{v \in V : f(v) = 0\}$$

Being the inverse image of a closed set by a continuous map, it is *closed*. It is easy to check that it is a k-subspace of V.

For a *closed* k-subspace H of a topological vectorspace V, we can form the *quotient* V/H as topological vectorspace, with k-linear quotient map $q : V \to V/H$ given as usual by

$$q : v \longrightarrow v + H$$

The **quotient topology** on E is the *finest* topology on E such that the quotient map $q : V \to E$ is continuous, namely, a subset E of V/H is open if and only if $q^{-1}(E)$ is open. It is easy to check that this is a topology.

[3.0.1] Remark: For *non*-closed subspaces H, the quotient topology on V/H is *not* Hausdorff. [5] Non-Hausdorff spaces are not topological vector spaces in our sense. For our purposes, we do not want non-Hausdorff spaces.

---

[5] That the quotient V/H by a not-closed subspace H is not Hausdorff is easy to see, using the definition of the quotient topology, as follows. Let v be in the closure of H but not in H. Then every neighborhood U of v meets H. Every neighborhood of $v+H$ in the quotient is of the form $v+H+U$ for some neighborhood U of v in V, and includes 0. That is, even though the image of v in the quotient is not 0, every neighborhood of that image includes 0.

Further, *unlike* general topological quotient maps,

[3.0.2] Proposition: For a closed subspace H of a topological vector space V, the quotient map $q : V \to V/H$ is *open*, that is, carries open sets to open sets.

Proof: Let U be open in V. Then

$$q^{-1}(q(U)) = q^{-1}(U + H) = U + H = \bigcup_{h \in H} h + U$$

This is a union of opens, so is open. ///

[3.0.3] Corollary: For a closed k-subspace W of a topological vectorspace V, the quotient V/W is a topological vectorspace. In particular, in the quotient topology points are closed.

Proof: The *algebraic* quotient exists without any topological hypotheses on W. Since W is closed, and since vector addition is a homeomorphism, $v + W$ is closed as well. Thus, its complement $V - (v + W)$ is open, so $q(V - (v + W))$ is open, by definition of the quotient topology. Thus, the complement

$$q(v) = v + W = q(v + W) = V/W - q(V - (v + W))$$

of the open set $q(V - (v + W))$ is closed. ///

[3.0.4] Corollary: Let $f : V \to X$ be a linear map with a closed subspace W of V contained in $\ker f$. Let $\bar{f}$ be the induced map $\bar{f} : V/W \to X$ defined by $\bar{f}(v + W) = f(v)$. Then f is continuous if and only if $\bar{f}$ is continuous.

Proof: Certainly if $\bar{f}$ is continuous then $f = \bar{f} \circ q$ is continuous. The converse follows from the fact that q is *open*. ///

That is, a continuous linear map $f : V \to X$ *factors through* any quotient V/W where W is a closed subspace contained in the kernel of f.

---

4. *More topological features*

Now we can consider the notions of **balanced subset, absorbing subset** and also **directed set, Cauchy net**, and *completeness*. We continue to suppose that the *scalars* k are a *non-discrete complete normed division ring*.

A subset E of V is **balanced** if for every $x \in k$ with $|x| \le 1$ we have $xE \subset E$.

Lemma: Let U be a neighborhood of 0 in a topological vectorspace V over k. Then U contains a *balanced* neighborhood N of 0.

Proof: By continuity of scalar multiplication, there is $\varepsilon > 0$ and a neighborhood U' of $0 \in V$ so that if $|x| < \varepsilon$ and $v \in U'$ then $xv \in U$. Since k is non-discrete, there is $x_o \in k$ with $0 < |x_o| < \varepsilon$. Since scalar multiplication by a non-zero element is a homeomorphism, $x_o U'$ is a neighborhood of 0 and $x_o U' \subset U$. Put

$$N = \bigcup_{|y| \le 1} y x_o U'$$

Then, for $|x| \le 1$, we have $|xy| \le |y| \le 1$, so

$$xN = \bigcup_{|y| \le 1} x(y x_o U') \subset \bigcup_{|y| \le 1} y x_o U' = N$$

This N is as desired. ///

A subset E of a (not necessarily topological) vectorspace V over k is **absorbing** if for every $v \in V$ there is $t_o \in \mathbf{R}$ so that $v \in \alpha E$ for every $\alpha \in k$ so that $|\alpha| \geq t_o$.

Lemma: Every neighborhood U of 0 in a topological vectorspace is *absorbing.*

Proof: We may as well shrink U so as to assure that U is balanced. By continuity of the map $k \to V$ given by $\alpha \to \alpha v$, there is $\varepsilon > 0$ so that $|\alpha| < \varepsilon$ implies that $\alpha v \in U$. By the non-discreteness of k, there is non-zero $\alpha \in k$ satisfying any such inequality. Then $v \in \alpha^{-1}U$, as desired. ///

Let S be a **poset**, that is, a set with a partial ordering $\geq$. We assume further that, given two elements $s, t \in S$, there is $z \in S$ so that $z \geq s$ and $z \geq t$. Then S is a **directed set**.

A **net** in V is a subset $\{x_s : s \in S\}$ of V indexed by a directed set S. A net $\{x_s : s \in S\}$ in a topological vectorspace V is a **Cauchy net** if, for every neighborhood U of 0 in V, there is an index s_o so that for $s, t \geq s_o$ we have $x_s - x_t \in U$. A net $\{x_s : s \in S\}$ is **convergent** if there is $x \in V$ so that, for every neighborhood U of 0 in V there is an index s_o so that for $s \geq s_o$ we have $x - x_s \in U$. Since points are closed, there can be *at most* one point to which a net converges. Thus, *a convergent net is Cauchy.* A topological vectorspace is **complete** if (also) every Cauchy net is convergent.

Lemma: Let Y be a vector subspace of a topological vector space X, and suppose that Y is *complete* when given the subspace topology from X. Then Y is a *closed* subset of X.

Proof: Let $x \in X$ be in the closure of Y. Let S be a local basis of opens at 0, where we take the partial ordering so that $U \geq U'$ if and only if $U \subset U'$. For each $U \in S$ choose

$$y_U \in (x + U) \cap Y$$

Then the net $\{y_U : U \in S\}$ converges to x, so is Cauchy. But then it must converge to a point in Y, so by uniqueness of limits of nets it must be that $x \in Y$. Thus, Y is closed. ///

[4.0.1] Remark: Unfortunately, *completeness* as above is too strong a condition for general topological vectorspaces, beyond Fréchet spaces. [6]

---

5. *Finite-dimensional spaces*

Now we look at the especially simple nature of finite-dimensional topological vectorspaces, and their interactions with other topological vectorspaces. [7] The point is that *there is only one topology on a finite-dimensional space.* This has important consequences.

[5.0.1] Proposition: For a one-dimensional topological vectorspace V, that is, a free module on one generator e, the map $k \to V$ by $x \to xe$ is a *homeomorphism.*

Proof: Since scalar multiplication is continuous, we need only show that the map is *open.* Given $\varepsilon > 0$, by the non-discreteness of k there is x_o in k so that $0 < |x_o| < \varepsilon$. Since V is Hausdorff, there is a neighborhood U of 0 so that $x_o e \notin U$. Shrink U so it is *balanced.* Take $x \in k$ so that $xe \in U$. If $|x| \geq |x_o|$ then $|x_o x^{-1}| \leq 1$, so that

$$x_o e \;=\; (x_o x^{-1})(xe) \in U$$

[6] A slightly weaker version of completeness, *quasi-completeness* or *local* completeness, *does* hold for most important natural spaces, and will be discussed later.

[7] We still only need suppose that the scalar field k is a complete non-discrete normed division ring.

by the balanced-ness of U, contradiction. Thus,

$$xe \in U \implies |x| < |x_o| < \varepsilon$$

This proves the claim. ///

[5.0.2] Corollary: Fix $x_o \in k$. A not-identically-zero k-linear k-valued function f on V is *continuous* if and only if the **affine hyperplane**

$$H = \{v \in V : f(v) = x_o\}$$

is *closed* in V.

Proof: Certainly if f is continuous then H is closed. For the converse, we need only consider the case $x_o = 0$, since translations (i.e., vector additions) are homeomorphisms of V to itself.

For v_o with $f(v_o) \neq 0$ and for any other $v \in V$

$$f(v - f(v)f(v_o)^{-1}v_o) = f(v) - f(v)f(v_o)^{-1}f(v_o) = 0$$

Thus, V/H is one-dimensional. Let $\bar{f} : V/H \to k$ be the induced k-linear map on V/H so that $f = \bar{f} \circ q$:

$$\bar{f}(v + H) = f(v)$$

Then $\bar{f}$ is a homeomorphism to k, by the previous result, so f is continuous. ///

In the following theorem, the three assertions are proven together by induction on dimension.

[5.0.3] Theorem:
- A *finite-dimensional* k-vectorspace V has just one topological vectorspace topology.
- A finite-dimensional k-subspace V of a topological k-vectorspace W is necessarily a *closed* subspace of W.
- A k-linear map $\phi : X \to V$ to a finite-dimensional space V is continuous if and only if the kernel is closed.

Proof: To prove the uniqueness of the topology, it suffices to prove that for any k-basis $e_1, \ldots, e_n$ for V, the map

$$k \times \ldots \times k \to V$$

given by

$$(x_1, \ldots, x_n) \to x_1e_1 + \ldots + x_ne_n$$

is a homeomorphism. Prove this by induction on the dimension n, that is, on the number of generators for V as a free k-module.

The case $n = 1$ was treated already. Granting this, we need only further note that, since k is complete, the lemma above asserting the closed-ness of complete subspaces shows that any one-dimensional subspace is necessarily closed.

Take $n > 1$. Let

$$H \ = \ ke_1 + \ldots + ke_{n-1}$$

By induction, H is closed in V, so the quotient V/H is a topological vector space. Let q be the quotient map. The space V/H is a one-dimensional topological vectorspace over k, with basis $q(e_n)$. By induction, the map

$$\phi : xq(e_n) = q(xe_n) \to x$$

is a homeomorphism to k.

Likewise, ke_n is a closed subspace and we have the quotient map

$$q' : V \to V/ke_n$$

We have a basis $q'(e_1), \ldots, q'(e_{n-1})$ for the image, and by induction the map

$$\phi' : x_1 q'(e_1) + \ldots + x_{n-1} q'(e_{n-1}) \to (x_1, \ldots, x_{n-1})$$

is a homeomorphism.

Invoking the induction hypothesis, the map

$$v \to (\phi \circ q)(v) \times (\phi' \circ q')(v)$$

is continuous to

$$k^{n-1} \times k \approx k^n$$

On the other hand, by the continuity of scalar multiplication and vector addition, the map

$$k^n \to V \quad \text{by} \quad x_1 \times \ldots \times x_n \to x_1 e_1 + \ldots + x_n e_n$$

is continuous. These two maps are mutual inverses, proving that we have a homeomorphism.

Thus, a n-dimensional subspace is homeomorphic to k^n, so is complete, since (as follows readily) a finite product of complete spaces is complete. Thus, by the lemma asserting the closed-ness of complete subspaces, it is closed.

Continuity of a linear map $f : X \to k^n$ implies that the kernel $N = \ker f$ is closed. On the other hand, if N is closed, then X/N is a topological vectorspace of dimension at most n. Therefore, the induced map $\bar{f} : X/N \to V$ is unavoidably continuous. But then $f = \bar{f} \circ q$ is continuous, where q is the quotient map. This completes the induction step. ///

---

6. *Convexity, seminorms, Minkowski functionals*

Now suppose that the scalar field k contains $\mathbb{R}$. Then the notion of **convexity** makes sense: a subset E of a vectorspace is *convex* when $tx + (1-t)y$ lies in E for all $0 \leq t \leq 1$ and for all $x, y \in E$.

The **Minkowski functional** μ_E on a vectorspace V attached to a set E in V is

$$\mu_E(v) \;=\; \inf\{t > 0 \;:\; v \in \alpha \cdot E \;\text{ for all }\; \backslash alf \in k \;\text{ with }\; |\alpha| \geq t, \;\} \qquad (\text{for } v \in V)$$

For E not *absorbing*, there might be v with *no* such t, necessitating that we put

$$\mu_E(v) \;=\; +\infty$$

Thus, E need not be absorbing to define these functionals, if infinite values are tolerated. [8]

[6.0.1] Proposition: For E *convex* and *balanced* in V,

$$\mu_E(x+y) \;\leq\; \mu_E(x) + \mu_E(y) \qquad (\text{for } x, y \in V)$$

$$\mu_E(tx) \;=\; t \cdot \mu_E(x) \qquad (\text{for } t \geq 0)$$

and

$$\mu_E(\alpha v) \;=\; |\alpha| \cdot \mu_E(v) \qquad (\text{for all } \alpha \in k)$$

---

[8] The value $+\infty$ cannot be treated as a number. As usual, $t + \infty = \infty$ for all real numbers t, but $\infty - \infty$ has no sensible value, and the sense of $0 \cdot \infty$ depends on circumstances.

Proof: As E is balanced,

$$\mu_E(\alpha v) \;=\; \inf\{t>0 \;:\; \alpha v \in \beta\cdot E \;\text{ for all }\; \beta\in k \;\text{ with }\; |\beta|\geq t,\;\}$$

$$=\; \inf\{t>0 \;:\; v \in \beta\cdot E \;\text{ for all }\; \beta\in k \;\text{ with }\; |\alpha\cdot\beta|\geq t,\;\}$$

Suppose $x\in sE$ for all $|s|\geq s_o$ and $y\in tE$ for all $|t|\geq t_o$. Then

$$|\frac{x}{s}|+|\frac{y}{t}|\leq 1$$

$$x+y\in sE+tE$$

for all $|s|\geq s_o$ and for all $|t|\geq t_o$.

---

7. *Countably normed, countably Hilbert spaces*

In practice, most Fréchet spaces have more structure than just the Fréchet structure: they are *projective limits of Hilbert spaces*, and even that in a rather special way. This type of additional information is exactly what is needed for several types of stronger results, concerning spectral theory, regularity results for differential operators, Schwartz-type kernel theorems, and so on.

The ideas here, although of considerable utility, are not made explicit as often as they merit. The present account is inspired by, and is partly an adaptation of, parts of the Gelfand-Shilov-Vilenkin-Graev monographs *Generalized Functions*. This material is meant to be a utilitarian substitute (following Gelfand *et alia*) for Grothendieck's somewhat more general concepts related to *nuclear spaces*.

Let V be a real or complex vectorspace with a collection of norms $|\,|_i$ for $i\in\mathbb{Z}$. We suppose that we have

$$\ldots\geq |v|_{-2}\geq |v|_{-1}\geq |v|_0\geq |v|_1\geq |v|_2\geq\ldots$$

for all $v\in V$. Let V_i be the *Banach space* obtained by taking the *completion* of V with respect to the i^{th} norm $|\,|_i$. The inequalities relating the various norms assure that for $i\leq j$ the identity map of V to itself induces (extending by continuity) *continuous inclusions*

$$\phi_{ij}: V_i\to V_j$$

Then it makes sense to take the *intersection* of all the spaces V_i: this is more properly described as an example of a *projective limit of Banach spaces*

$$\bigcap_i V_i = \operatorname{proj\,lim}_i V_i$$

It is clear that V is contained in this intersection (certainly in the sense that there is a natural injection, and so on). If the intersection is *exactly* V then V is a **countably normed space** or **countably Banach space**.

This situation can also arise when we have positive-definite hermitian inner products $\langle,\rangle_i$ with $i\in\mathbb{Z}$. Let $|\,|_i$ be the norm associated to $\langle,\rangle_i$. Again suppose that

$$|v|_i\geq |v|_{i+1}$$

for all $v\in V$ and for all indices i. If the intersection

$$\bigcap_i V_i = \operatorname{proj\,lim}_i V_i\supset V$$

is *exactly* V then we say that V is a **countably Hilbert space**.

[7.0.1] Remark: The notion of countably Hilbert space is worthwhile only for real or complex scalars, while the countably Banach concept has significant content over more general scalar fields.

[7.0.2] Remark: We can certainly take projective limits over more complicated indexing sets. And we can take $| \, |_i = | \, |_{i+1}$ for $i \geq 0$ if we want to focus our attention only on the 'negatively indexed' norms or inner products.

---

8. *Local countability*

For any algebraic subspace Y of the dual space V^* of continuous linear functionals on V, *if* Y *separates points on* V we can form the Y**-(weak-)topology** on V by taking seminorms

$$\nu_\lambda(v) := |\lambda(v)|$$

for $\lambda \in Y$.

The assumption that Y separates points is necessary to assure that the topology attached to this collection of semi-norms is such that *points are closed*. For example, if V is locally convex and Y is all of V^*, then this separation property is assured by the Hahn-Banach theorem.

If Y separates points on V and if V is not a countable union of finite-dimensional subspaces, then the Y-topology on V cannot have a countable local basis.

For example, if V is an infinite-dimensional Frechet space, then (from Baire's theorem) its dual is not locally countable.

Proof: Given $y \in Y$ and $\varepsilon > 0$, suppose that there are $y_1, \ldots, y_n \in V$ and $\varepsilon_1, \ldots, \varepsilon_n > 0$ so that for $v \in V$ we have

$$|y_i(v)| < \varepsilon_i \ \forall i \ \to |y(v)| < \varepsilon$$

If so, then certainly $y_i(v) = 0$ for all i would imply that $|y(v)| < \varepsilon$. Let H denote the closed subspace of $v \in V$ where $y_i(v) = 0$ for all i. Then $|y(v)| < \varepsilon$ on H implies that $y(v) = 0$ on H.

We claim that then y is a linear combination of the y_i. Without loss of generality we may suppose that the y_i are linearly independent. Consider the quotient map $q : V {\to} V/H$. From elementary linear algebra, without any topological consideration, that the quotient V/H is n-dimensional, and has dual space spanned by the functionals

$$\bar{y}_i(v + H) = y_i(v)$$

The functional $y(v)$ induces a continuous functional

$$\bar{y} : V/H \to \mathbf{C}$$

since y vanishes on H. Thus, $\bar{y}$ is a linear combination of the $\bar{y}_i$.

The fact that $\bar{y}$ is a linear combination of the $\bar{y}_i$ implies that v is the corresponding linear combination of the v_i.

This shows that, if it were the case that V had a countable basis in the Y-topology, then there would be countably-many y_i so that every vector in Y would be a *finite* linear combination of the y_i. ///