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Catalogue of Useful Topological Vectorspaces

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We take for granted that Hilbert spaces and Banach spaces are understood. **Frechet spaces** are projective limits of countable collections of Banach spaces. Thus, they are (complete) metrizable, and are *locally convex.*

We use standard *multi-index* notation: for $i = (i_1, \ldots, i_n) \in \mathbf{Z}^n$, write

$$|i| = \sum_j i_j$$

For $|i| \leq k$ and for a k-times continuously differentiable function f on $\mathbf{R}^n$, write

$$f^{(i)} = \frac{\partial^{|i|} f}{\partial x_1 \ldots \partial x_n}$$

A function f is *rapidly decreasing* if $f(x)|x|^t$ goes to zero as $|x| \to +\infty$, for *all* $t \geq 0$).

The space C^k_K of k-times continuously differentiable functions on $\mathbf{R}^n$ with support in K is a *Banach space* when given the norm

$$|f| = \sum_{|i| \leq k} \sup_{x \in K} |f^{(i)}(x)|$$

(Note that this space of functions is quite a bit smaller than the space $C^k(K)$ of k-times differentiable functions on K, since the definition of C^k_K requires that functions and their derivatives go to zero at the boundary of K, and so on).

The space $\mathcal{D}_K = C^\infty_K$ of smooth functions on $\mathbf{R}^n$ with support in compact K is a *Frechet* space, being the projective limit (intersection) of the Banach spaces C^k_K.

The space $\mathcal{D}$ of **test functions** on $\mathbf{R}^n$ is the collection of all compactly-supported smooth (i.e., infinitely-differentiable) functions on $\mathbf{R}^n$. It is the *direct limit* (ascending union) of the Frechet spaces $\mathcal{D}_K$ of smooth functions with support in a given compact set K. It is locally convex, but *not* metrizable: it is a countable union of nowhere-dense subsets, namely the $\mathcal{D}_K$ as K ranges over a countable collection of compacta whose union is all of $\mathbf{R}^n$.

The **Schwartz space** $\mathcal{S}$ on $\mathbf{R}^n$ is the collection of smooth functions all whose partial derivatives are *rapidly decreasing.*

This is a Frechet space (countable projective limit of Banach spaces). Specifically, we have the *countable* collection of seminorms

$$\nu_{k,\ell}(f) = \sup_{x \in \mathbf{R}^n, |i| \leq \ell} (1 + |x|)^k \, |f^{(i)}(x)|$$

for non-negative integers k, ℓ.

The space $\mathcal{E}$ of *all* smooth functions on $\mathbf{R}^n$ is a Frechet space (countable projective limit of Banach spaces), with seminorms

$$\nu_{k,K}(f) = \sup_{x\in K, |i|\le k} |f^{(i)}(x)|$$

as K ranges over compact subsets of $\mathbf{R}^n$ and k ranges over positive integers.

Since we can write $\mathbf{R}^n$ as a countable union of compacta, there is a *cofinal countable* collection of these seminorms, entitling us to say that the projective limit is 'countable'.

The space $\mathcal{D}^* = \mathcal{D}'$ of all **distributions** on $\mathbf{R}^n$ is the continuous linear dual of the space $\mathcal{D}$ of test functions. It is usually given the *weak star-topology.*

The space $\mathcal{S}^* = \mathcal{S}'$ is the space of **tempered distributions**. This is the continuous linear dual to the space of Schwartz functions. It is usually given the *weak star-topology.*

The space $\mathcal{E}^* = \mathcal{E}'$ is the space of **compactly-supported distributions**. It is the continuous linear dual to the space of all smooth functions. It is usually given the *weak star-topology.*

The continuous injections

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}$$

gives rise to continuous maps

$$\mathcal{E}^* \to \mathcal{S}^* \to \mathcal{D}^*$$

Since in $\mathcal{D} \subset \mathcal{S} \subset \mathcal{E}$ each has dense image, the maps of dual spaces are *injective*, but not surjective.

One version of the **Sobolev space** L^p_k is that it is the Banach space with norm

$$|f|_{L^p_k} = \sup_{|i|\le k} |f^{(i)}|_p$$

where the norm $||_p$ is the usual L^p-norm.

There is a certain possibility of confusion here: unlike the C^k-topologies just discussed, we do *not* know that L^p_k-limits of genuinely differentiable functions are again genuinely differentiable. The functions in L^p_k are said to be **weakly** k-times differentiable, to distinguish this notion of differentiability from the 'genuine' one. Nevertheless, for a function to possess a weak derivative is a *stronger* property than that of possessing a mere *distributional* derivative.

So there are two possible ways to give a careful definition of these Sobolev spaces, one 'from *inside'*, the other 'from *outside'*.

The first is to say that L^p_k is the *completion* of $\mathcal{D}$ with respect to the L^p_k-norm.

The second is to say that L^p_k is the collection of all *distributions* u so that all derivatives of u of order $\le k$ are (given by) L^p functions.

The space of **locally** L^p functions on $\mathbf{R}^n$ (with $1 \le p < \infty$) denoted $L^p_{\text{loc}} = L^p_{\text{loc}}(\mathbf{R}^n)$, is the *direct limit* of the Banach spaces $L^p(K)$ as K runs over compact subsets of $\mathbf{R}^n$. Since $\mathbf{R}^n$ is a countable union of compacta, this is a *countable* direct limit.

For $0 < p < 1$, and for measure space (X, μ), the space $L^p(X, \mu)$ of functions f so that

$$\int_X |f(x)|^p \, d\mu(x) < \infty$$

is a *not locally convex* topological vectorspace, complete with respect to the (translation-invariant) metric

$$d(f,g) = \int_X |f(x) - g(x)|^p \, d\mu(x) < \infty$$

Note that there is not $1/p^{\text{th}}$ power taken at the end, in contrast to the case $1 \le p < \infty$.

For $\varphi \in \mathcal{S}$ with $\int \varphi \neq 0$, for $t > 0$, let

$$\varphi_t(x) = t^n \varphi(tx)$$

For a tempered distribution u, define a function $M_\varphi u$ on $\mathbf{R}^n$ by

$$(M_\varphi u)(x) = \sup_{t>0} |\varphi_t * u(x)|$$

Generally, for $0 < p \le \infty$, define the **Hardy space**

$$H^p = H^p(\mathbf{R}^n) = \{u \in \mathcal{S}^* : \text{ for some } \varphi \in \mathcal{S},\ M_\varphi u \in L^p\}$$

The *Hardy spaces* H^p, with $0 < p \le 1$ in some sense replace the L^p spaces for $p < 1$. The analogous definitions for $p > 1$ provably give $L^p = H^p$, while this is not at all so for $p \le 1$. For $p = 1$, we have $H^1 \subset L^1$, but equality does not hold.

Although much more can be proven, all that is immediately visible from the definition is that H^p is a direct limit of Frechet spaces. The operator M_φ is a kind of **maximal operator**. The **standard maximal operator** M on functions on $\mathbf{R}^n$ is defined by

$$(Mf)(x) = c_n \cdot \sup_{t>0} \frac{1}{t^n} \int_{|y-x| \le t} |f(y)| \, dy$$

The **Lipschitz** or **Holder** spaces Λ_γ for $0 < \gamma < 1$ consist of the functions f so that there is a constant $A = |f|_{\Lambda_\gamma}$ so that

$$|f(x)| \le A \qquad \text{almost everywhere}$$

and also for each y

$$\sup_x |f(x+y) - f(x)| \le A \cdot |y|^\gamma$$

This is a *Banach space*.
