(*January 19, 2013*)

Measurable Choice Functions

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/fun/choice_functions.pdf]

This note is essentially a rewrite of Appendix V of J. Dixmier's *von Neumann Algebras*, North Holland, 1981.

The point is to obtain the practical special case:

[0.0.1] Corollary: Let X be a locally compact Hausdorff second-countable [1] topological space with a complete positive Borel measure ν giving compacta finite measure. Let Y be a complete second-countable metric space, Γ a Borel subset of $X \times Y$. Let X_o be the projection of Γ to X. [2] Then there is a ν-measurable function $f : X_o \to Y$ whose graph lies inside Γ. ///

Using terms defined just below which are convenient to indicate the *sharp* hypotheses for the result, the theorem itself is

[0.0.2] Theorem: Let X and Y be polish spaces and Γ a Souslin subset of $X \times Y$. Let X_o be the projection of Γ to X (so is Souslin). Then there is a *weakly* Souslin map $f : X_o \to Y$ whose graph is contained in Γ.

It is best to arrange things properly prior to the proof.

A topological space is **polish** if it is a second-countable (i.e., it has a countable basis) complete metric [3] space. A countable product of polish spaces X_n with metrics d_n is polish, with metric [4]

$$d(\{x_n\}, \{y_n\}) = \sum_{n \geq 1} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

A countable disjoint union [5] of polish spaces is polish, with metric

$$d(x, y) = \begin{cases} 1 & \text{(for } x, y \text{ in distinct spaces in the union)} \\ d_n(x, y) & \text{(for } x, y \text{ in the } n^{th} \text{ space in the union)} \end{cases}$$

A closed subset of a polish space is polish. Slightly surprisingly, an *open* subset of a polish space is polish: let F be the complement of the open subset U of a polish space X with metric d, and define a metric δ on U by

$$\delta(x, y) = d(x, y) + \left| d(x, F)^{-1} - d(y, F)^{-1} \right|$$

---

[1] This is the usual not-so-intuitive way to say that X has a countable basis.

[2] It follows that X_o is ν-measurable. See later discussion.

[3] As may become apparent along the way, one might distinguish *metric* spaces from *metrizable*. For example, many different metrics may give the same topology. But this is not at all the main point here.

[4] There are many other similarly-defined metrics that give a homeomorphic topology on the product. In particular, we acknowledge that this metric is very much not canonical. At the same time, creating a formalism concerning equivalence classes of metrics is not a high priority at this moment, so we will not do it.

[5] The countability of the union is necessary for that union to still have a countable basis. The metrizability of the union does not depend upon the countability.

The space $\mathbb{N}$ of positive integers with the usual metric inherited from the real numbers $\mathbb{R}$ is polish. Then the countable product $\mathbb{N}^{\mathbb{N}}$ with metric (as above)

$$d(\{m_i\},\{n_i\}) = \sum_{i\geq 1} 2^{-i} \, \frac{|m_i - n_i|}{1+|m_i-n_i|}$$

is polish.

For X a polish space, there is a continuous surjection $\mathbb{N}^{\mathbb{N}} \to X$, given as follows. Given $\varepsilon > 0$ there is a countable covering of X by closed sets of diameter less than ε. From this one may contrive [6] a map F from finite sequences $n_1, \ldots, n_k$ in $\mathbb{N}$ to closed sets $F(n_1, \ldots, n_k)$ in X such that

- $F(\phi) = X$
- $F(n_1, \ldots, n_k) = F(n_1, \ldots, n_k, 1) \cup F(n_1, \ldots, n_k, 2) \cup F(n_1, \ldots, n_k, 3) \cup \ldots$
- The diameter of $F(n_1, \ldots, n_k)$ is less than 2^{-k}.

Then for $y = \{n_i\} \in \mathbb{N}^{\mathbb{N}}$ the sequence $E_i = F(n_1, \ldots, n_k)$ is a nested sequence of closed subsets of X with diameters less than 2^{-k}, respectively. Thus, $\bigcap_i E_i$ consists of a single point of X. On the other hand, every $x \in X$ lies inside some $\bigcap_i E_i$. Continuity is straightforward to verify.

A second-countable locally-compact Hausdorff [7] space is polish: [8] let U_i be a countable basis of opens with compact closures K_i, and let V_i be open with compact closure and containing K_i. From Urysohn's Lemma, let $0 \leq f_i \leq 1$ be continuous functions identically 0 off V_i, identically 1 on K_i, and put

$$d(x,y) \;\;=\;\; \sum_i 2^{-i} \Big| f_i(x) - f_i(y) \Big| \;+\; \Big| \frac{1}{\sum_i 2^{-i} f_i(x)} - \frac{1}{\sum_i 2^{-i} f_i(y)} \Big|$$

The triangle inequality for the usual absolute value shows that this is a metric. This metric gives the same topology, and it is straightforward to verify its completeness, once that peculiar last term is added, which prevents seemingly-Cauchy sequences from escaping to infinity.

A **Souslin** set is a continuous image of a polish space in another polish space.

Countable unions of Souslin sets are Souslin: for $f_n : X_n \to Y$ a countable collection of continuous maps from polish X_n to polish Y, then the disjoint union of the X_n is polish, and the obvious map [9]

$$f(x) = f_n(x) \;\; (\text{for } x \in X_n)$$

from the disjoint union to Y is continuous, realizing the countable union of the images as a continuous image of a polish space.

[6] There are many such maps F. In particular, one should not hope for canonicalness.

[7] The hypothesis of Hausdorff-ness is too strong, though is easier to understand than the *correct* hypothesis. That is, *Urysohn's Metrization Theorem* (as in J. Kelley, *General Topology*, Van Nostrand, 1955), shows that a *regular T_1-space with a countable basis is metrizable.* Recall that *regular* means that for each x in the space and for each open neighborhood U of x there is a neighborhood V of x such that the closure of V is inside U. The T_1 hypothesis is that all singleton sets $\{x\}$ are closed. We are not presently concerned with implications among these hypotheses.

[8] Thanks to Jan van Casteren for twice pointing out flaws in previous versions, and indicating repairs. On the third try, I hope things are correct! He also noted a useful reference: T. Kechris, *Classical descriptive set theory*, Graduate Texts in Math. 156, Springer-Verlag, 1995. Theorem 5.3, page 29, proves equivalences between several related notions.

[9] From a slightly more categorical viewpoint, this observation is about *coproducts.*

Countable intersections of Souslin sets are Souslin: let $f_n : X_n \to Y$ be continuous maps of polish spaces, let X be the cartesian product of the X_n, and define

$$Q = \{\{x_n\} \in X : f_m(x_m) = f_n(x_n) \text{ for all } m, n\}$$

Since Q is a closed subspace of a polish space, it is polish. The It continuously surjects to the intersection of the images $f_n(X_n)$ since the map $f : Q \to Y$ defined by

$$f(\{x_n\}) = f_1(x_1)$$

is well-defined.

Since closed and open subsets of polish spaces are polish, and since inclusion maps are continuous, both open and closed subsets of polish spaces are polish. In particular, the set of subsets E of a polish space X such that E is Souslin *and* the complement $X - E$ is Souslin is a σ-algebra containing the Borel sets. [10] In particular, in polish spaces *Borel* implies *Souslin.*

A countable intersection of countable unions of compact sets is called a $K_{\sigma\delta}$-set.

Given a pre-compact [11] set E in a polish space Y, there is a *compact* polish space X, a $K_{\sigma\delta}$-set X_o in X, and continuous $f : X \to Y$ such that $f(X_o) = E$. Indeed, we may suppose that Y is the closure of E, so is compact. Let $\varphi : \mathbb{N}^{\mathbb{N}} \to E$ be a continuous surjection, as described above. Let $\tilde{\mathbb{N}}$ be the one-point compactification of $\mathbb{N}$. Then $\mathbb{N}^{\mathbb{N}}$ is the intersection of a sequence of opens U_i in $\tilde{\mathbb{N}}^{\mathbb{N}}$. Let $X = \tilde{\mathbb{N}}^{\mathbb{N}} \times Y$ and $f : X \to Y$ the projection. Let X_o be the graph of φ in $\mathbb{N}^{\mathbb{N}} \times Y \subset \tilde{\mathbb{N}}^{\mathbb{N}} \times Y$. Certainly $f(X_o) = E$. Since φ is continuous, X_o is closed [12] in $\mathbb{N}^{\mathbb{N}} \times Y$.

The closure $\overline{X_o}$ of X_o in X is compact, since X is compact and Hausdorff. Then

$$X_o = \overline{X_o} \cap (\mathbb{N}^{\mathbb{N}} \times Y)$$

$$\mathbb{N}^{\mathbb{N}} \times Y = (\bigcap_i U_i) \times Y = \bigcap_i (U_i \times Y)$$

An open U_i in $\tilde{\mathbb{N}}^{\mathbb{N}}$ is a countable union of closed (hence compact) subsets of $\tilde{\mathbb{N}}^{\mathbb{N}}$, so $U_i \times Y$ is indeed a countable union of compact subsets of X. ///

[0.0.3] Proposition: In a locally compact Hausdorff second-countable (hence, polish) space Y, given a positive Borel measure ν, a Souslin set E in X is ν-measurable with respect to the *completion* of ν. That is, given $\varepsilon > 0$ there are open U and compact K such that $K \subset E \subset U$ and $\nu(U) - \nu(K) < \varepsilon$, and, further, if the *outer measure*

$$\nu^*(E) = \inf_{\text{open U}\supset\text{E}} \nu(U)$$

is $\nu^*(E) = +\infty$, then the *inner measure*

$$\nu_*(E) = \sup_{\text{compact K}\subset\text{E}} \nu(K)$$

---

[10] The set of Borel sets in a topological space X is, by definition, the smallest σ-algebra containing the open sets in X.

[11] Here *pre-compact* means *having compact closure.* By contrast, the precise import of the term in non-metrizable spaces can be more complicated.

[12] With Hausdorff spaces A, B and a map $f : A \to B$, continuity of f can readily be shown to be equivalent to closedness of the graph of f.

is $\nu_*(E) = +\infty$ as well.

Proof: Let X be compact, X_o a $K_{\sigma\delta}$-set in X, and $f : X \to Y$ continuous such that $f(X_o) = E$. Let

$$X_o = (C_{11} \cup C_{12} \cup C_{13} \cup \ldots) \cap (C_{21} \cup C_{22} \cup C_{23} \cup \ldots) \cap \ldots$$

be an expression for X_o where all the C_{ij} are compact. Without loss of generality, $C_{ij} \subset C_{i,\mathbf{j}+1}$ for all i, j. Take $t < \nu^*(E)$. We will find a compact $X_1 \subset X_o$ such that $\nu(f(X_1)) \geq t$. Since $f(X_1)$ is compact, this will suffice to prove the proposition.

To this end, we claim that there is a sequence $\{n_i\}$ of integers such that the sets

$$D_k = X_o \cap C_{1,n_1} \cap C_{2,n_2} \cap \ldots \cap C_{k,n_k}$$

have the property that $\nu^*(f(D_k)) > t$ for all k. Indeed, first take n_1 sufficiently large such that $X_o \cap C_{1,n_1}$ has measure above t. Then, for the induction step, given n_i for $i < \ell$, since

$$D_{\ell-1} \subset X_o \subset C_{\ell 1} \cup C_{\ell 2} \cup C_{\ell 3} \cup \ldots$$

we have

$$D_{\ell-1} = (D_{\ell-1} \cap C_{\ell 1}) \cup (D_{\ell-1} \cap C_{\ell 2}) \cup (D_{\ell-1} \cap C_{\ell 3}) \cup \ldots$$

Since $C_{\ell j} \subset C_{\ell j+1}$, as $i \to \infty$,

$$\nu^*(f(D_{\ell-1} \cap C_{\ell i})) \to \nu^*(f(D_{\ell-1}))$$

Thus, for n_ℓ sufficiently large,

$$\nu^*(f(D_{\ell-1} \cap C_{\ell i})) > t$$

This proves the claim

To prove the proposition, let

$$X_1 = D_1 \cap D_2 \cap \ldots = C_{1n_1} \cap C_{2n_2} \cap \ldots$$

Since the partial intersections

$$C_{1n_1} \cap C_{2n_2} \cap \ldots \cap C_{kn_k}$$

form a decreasing sequence of compacts with intersection X_1, that intersection X_1 is compact, and $f(X_1)$ is compact, and

$$f(X_1) = \bigcap_k f(C_{1n_1} \cap C_{2n_2} \cap \ldots \cap C_{kn_k})$$

Thus,

$$\nu(f(X_1)) = \lim_k \nu\left(f(C_{1n_1} \cap C_{2n_2} \cap \ldots \cap C_{kn_k})\right) \geq \lim_k \nu^*(f(D_k)) \geq t$$

by the construction. Thus, Souslin sets are measurable. ///

[0.0.4] Remark: In second-countable locally compact Hausdorff spaces inner and outer measures associated to ν coincide with ν on opens and compacts, *if* all compacts have finite measure. *We do assume that compacta have finite measure.*

[0.0.5] Proposition: Totally order $\mathbb{N}^{\mathbb{N}}$ lexicographically. Then every *closed* subset E of $\mathbb{N}^{\mathbb{N}}$ has a least element.

Proof: Let n_1 be least in $\mathbb{N}$ such that there is $x = (n_1, \ldots)$ in E. Let n_2 be the least in $\mathbb{N}$ such that there is $x = (n_1, n_2, \ldots)$ in E, and so on. Choosing the n_i inductively, let $x_o = (n_1, n_2, n_3, \ldots)$. This x_o satisfies $x_o \leq x$ in the lexicographic ordering for every $x \in E$, and x_o is in the closure of E in the metric topology introduced earlier. ///

Now let Q be the subset of $\mathbb{N}^\mathbb{N}$ consisting of sequences with only finitely-many entries > 1. For $x, y \in \mathbb{N}^\mathbb{N}$ put

$$[x, y) = \{z \in \mathbb{N}^\mathbb{N} : x \le z < y\}$$

$$(*, y) = \{z \in \mathbb{N}^\mathbb{N} : z < y\}$$

We claim that every open U in $\mathbb{N}^\mathbb{N}$ is a (countable) union of sets $[x, y)$ with x, y. Indeed, given $z = (z_1, z_2, \ldots)$ in U, there is an integer k such that if $w = (w_1, w_2, \ldots)$ has $w_i = z_i$ for $1 \le i \le k$ then $w \in U$. Letting

$$x = (z_1, z_2, \ldots, z_k, 1, 1, 1, \ldots)$$

$$y = (z_1, z_2, \ldots, z_k, z_{k+1}, 1, 1, \ldots)$$

proves the claim. ///

Let X be a polish space, Y a topological space, X_o a subset of X, and $f : X_o \to Y$ a map of sets. The map f is **Souslin** if

$$f^{-1}(\text{open set in } Y) = (\text{Souslin set in } X) \cap X_o$$

Let S be the σ-algebra generated by Souslin sets in X. A map of sets $f : X_o \to Y$ is **weakly Souslin** if, for every open U in Y the inverse image $f^{-1}(U)$ is of the form

$$f^{-1}(U) = X_o \cap E$$

for some $E \in S$.

From above, for X locally compact Hausdorff and second-countable, S consists of ν-measurable [13] sets for any positive Borel measure ν giving compacta finite measure. Thus, for such X, an assumption that $f : X_o \to Y$ is weakly Souslin implies that f is ν-measurable.

Finally:

Proof: *(of Theorem)* Let $\varphi : \mathbb{N}^\mathbb{N} \to \Gamma$ be a continuous surjection, with pr_X and pr_Y the projections of $X \times Y$ to X and Y, respectively. Since $\mathrm{pr}_X \circ \varphi$ is continuous, the inverse image $(\mathrm{pr}_X \circ \varphi)^{-1}(x_o)$ is closed $\mathbb{N}^\mathbb{N}$ for any x_o, so (from above) has a least element $\lambda(x_o)$ in the lexicographic ordering. The graph of $f = \mathrm{pr}_Y \circ \varphi \circ \lambda$ is a subset of Γ. As $\mathrm{pr}_Y \circ \varphi$ is continuous, to prove f weakly Souslin it suffices to show that λ is weakly Souslin.

To this end, note that, from above, any open in $\mathbb{N}^\mathbb{N}$ is a *countable* union of sets $[z, w)$. From the definition of λ,

$$\lambda^{-1}([z, w)) = (\mathrm{pr}_X \circ \varphi)((*, w)) - (\mathrm{pr}_X \circ \varphi)((*, z))$$

Each $(*, w)$ is open in the polish space $\mathbb{N}^\mathbb{N}$, so is polish, and its continuous image $(\mathrm{pr}_X \circ \varphi)((*, w))$ is Souslin, by definition. The difference of two such sets is in the σ-algebra in X generated by Souslin sets, as are countable unions of such. Thus, λ is weakly Souslin, so f is, as well. ///

---

[13] As earlier, this measurability means measurable with respect to the *completion* of ν.