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Compact families of open, closed sets

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

By definition, *finite* intersections of open sets are open, and *finite* unions of closed sets are closed. Typically, *infinite* intersections of open sets are not open, and *infinite* unions of closed sets are not closed.

Nevertheless, *compact* objects mimic *finite*:

[0.0.1] Proposition: Let G be either a (locally compact, Hausdorff) topological group, or a (Hausdorff) topological vector space, acting [1] continuously on a topological space X. For a compact subset Θ of G, a closed subset C of X, and an open subset D of X,

$$\bigcap_{g\in\Theta} g\cdot D \;=\; \text{open subset of } X \qquad\qquad \bigcup_{g\in\Theta} g\cdot C \;=\; \text{closed subset of } X$$

[0.0.2] Remark: That is, **compact families** of open sets have open intersection, and **compact families** of closed sets have closed union.

Proof: The two statements are equivalent, so we prove that unions of compact families of closed sets are closed.

For a point x of X not in $\Theta\cdot C$, we find an open set containing x and not meeting $\Theta\cdot C$. Certainly $\Theta^{-1}x\cap C=\phi$. For each $y\in\Theta^{-1}x$, by the continuity at $e\times y$ of the action, there is a neighborhood U_y of e in G and V_y a neighborhood of y in X such that

$$U_y\cdot V_y \;\cap\; C \;=\; \phi$$

Since inversion is continuous on G, Θ^{-1} is compact. For each $g\in\Theta^{-1}$, the set $U_{gx}\cdot g$ is an open neighborhood of g, so by compactness of Θ^{-1} there is a finite subcover

$$U_{g_1x}\cdot g_1 \;\cup\;\ldots\;\cup\; U_{g_nx}\cdot g_n \;\supset\; \Theta^{-1}$$

The open set

$$V \;=\; \bigcap_i V_{g_ix}$$

contains x. For $g\in\Theta^{-1}$, there is at least one index i so that $g\in U_{g_ix}\,g_i$. Then

$$g\cdot V \;\subset\; U_{g_ix}\,g_i\cdot V_{g_ix} \;\subset\; X-C$$

Thus,

$$\Theta^{-1}\cdot V \;\subset\; X-C$$

or

$$V \;\cap\; \Theta\cdot C \;=\; \phi$$

as desired. ///

[1] A continuous group action of G on X is a continuous map

$$G\times X\longrightarrow X \qquad \text{by} \qquad g\times x\longrightarrow gx \qquad (\text{for } g\in G \text{ and } x\in X)$$

Associativity is required: $g(hx)=(gh)x$ for $g,h\in G$ and $x\in X$. The identity element $e\in G$ is required to act trivially: $ex=x$, for all $x\in X$.