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Functional analysis exercises-discussion 02

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Was due Wed, 24 Oct 2012, preferably as PDF emailed to me.

[02.1] *Convincingly* and *not-ugly-ly* prove that e^{-1/x^2} (naturally extended by 0 at $x = 0$) is infinitely differentiable at 0.

Using $e^{-a} = 1/e^a$,

$$e^{-1/x^2} \;=\; \frac{1}{e^{1/x^2}} \;=\; \frac{1}{1+\frac{1}{x^2}+\ldots+\frac{1}{\ell!\cdot x^{2\ell}}+\ldots} \;\le\; \frac{1}{\frac{1}{\ell!\cdot x^{2\ell}}} \;=\; \ell!\cdot x^{2\ell} \qquad \text{(for every } 1\le \ell\in\mathbb{Z})$$

A trivial induction step shows that (at $x \neq 0$) $\frac{d^n}{dx^n}e^{-1/x^2}$ is of the form $R(x)\cdot e^{-1/x^2}$ for some rational function R depending on n. Thus, with these derivatives inductively extending to $x = 0$ by taking value 0 there,

$$\Big|\frac{R(x)e^{-1/x^2} \;-\; 0}{x}\Big| \;\le\; \Big|\frac{R(x)\cdot \ell!\cdot x^{2\ell}}{x}\Big| \;\longrightarrow\; 0 \qquad \text{(as } x\to 0, \text{ for } \ell \text{ large enough)}$$

Thus, the $(n+1)$st derivative is also 0 at 0, and the induction succeeds. ///

[02.2] Show that for $0 \le c_n \in \mathbb{R}$ with c_n decreasing *monotonicly* to 0, the Fourier series $\sum_n c_n\, e^{inx}$ *converges* at $x \notin 2\pi\mathbb{Z}$, although not necessarily *absolutely*.

Yes, this is a special case of *summation by parts*, since all we really need is that the partial sums $\sum_{|n|\le N} e^{inx}$ be *bounded*, which holds for fixed $x \notin 2\pi\mathbb{Z}$, since that finite sum a geometric series, which sums to $(e^{-inx} - e^{i(n+1)x})/(1 - e^{ix})$.

Suppose, generally, that the finite subsums $B_N = \sum_{|n|\le N} b_n$ are *bounded*, and c_n's positive real, decreasing monotonically to 0. Let's break the sum into two pieces, one with $n \ge 0$ and the other with $n < 0$, to avoid clumsiness of language. We estimate the tails of the sum: partial summation gives

$$\sum_{M\le n\le N} b_n c_n \;=\; b_M c_M + \sum_{M+1\le \ell\le N} (B_\ell - B_{\ell-1})\cdot c_\ell \;=\; b_M c_M + \sum_{M+1\le \ell\le N-1} B_\ell\cdot (c_\ell - c_{\ell+1}) + B_N c_N$$

The first and last terms $b_M c_M$ and $B_N c_N$ go to 0, because b_M and B_N are bounded, and c_M and c_N go to 0. Letting B be a finite bound for the B_N's, the middle sum is estimated by

$$\Big|\sum_{M+1\le \ell\le N-1} B_\ell\cdot (c_\ell - c_{\ell+1})\Big| \;\le\; B\cdot \sum_{M+1\le \ell\le N-1} |c_\ell - c_{\ell+1}| \;=\; B\cdot \sum_{M+1\le \ell\le N-1} c_\ell - c_{\ell+1} \;=\; B\cdot \big(c_{M+1} - c_N\big)$$

using the monotonicity of the c_ℓ's. This, too, goes to 0. That is, the finite tails go to 0, so the infinite sum converges. ///

[02.3] The Fejér kernel discussion was carried out in considerable detail in class, I think.

[02.4] Map $\mathbb{C}^n \to \mathbb{C}^{n+1}$ by $(z_1,\ldots,z_n) \to (z_1,\ldots,z_n,0)$. Let $V = \mathrm{colim}_n\mathbb{C}^n = \bigcup_n \mathbb{C}^n$, with the colimit topology, in which a basis of opens at 0 is given by convex hulls of unions $B = \bigcup_n B_n$ where B_n is an open ball of some positive radius, at 0, in $\mathbb{C}^n$. Show that V violates the conclusion of the Baire category theorem, so is not complete-metrizable. Here *Cauchy sequences* $\{x_n\}$ are those such that, given a neighborhood N of 0, there is n_o such that $x_m - x_n \in N$ for all $m, n \geq n_o$. Show that Cauchy sequences *converge*.

Each subspace $\mathbb{C}^n$ is nowhere dense in the colimit, because elements of $\mathbb{C}^n$ cannot come close to elements of $\mathbb{C}^N$ with $N > n$ and non-zero components at $(n+1)$st and subsequent entries. Thus, $\mathrm{colim}_n \mathbb{C}^n$ is a countable union of nowhere dense subsets, violating the conclusion of the Baire category theorem. Thus, this colimit is not complete-metrizable.

We claim that in the colimit (*diamond*) topology, every Cauchy sequence in fact lies in some limitand $\mathbb{C}^n$, the latter with its usual complete topology, so the sequence converges.

Indeed, let $v_1, v_2, \ldots$ be a sequence *not* lying in any single limitand $\mathbb{C}^n$. Without loss of generality, replace the sequence by a subsequence so that $v_\ell \in C^{n_\ell}$ but *not* in $\mathbb{C}^{n_\ell - 1}$, with a strictly increasing sequence $n_1 < n_2 < n_3 < \ldots$. Let

$$v_\ell \;=\; (v_{\ell 1}, v_{\ell 2}, \ldots, v_{\ell n_\ell}, 0, 0, \ldots)$$

Note that $v_{\ell n_\ell} \neq 0$. Let

$$B_\ell \;=\; \text{open ball of radius } \min_{j \leq \ell} \frac{|v_{j n_j}|}{2} \;\subset\; \mathbb{C}^{n_\ell}$$

Then $v_\ell - v_{\ell-1}$ has n_ℓ^{th} component $v_{\ell n_\ell}$, so is *not* in B_ℓ. Let B be the convex hull of the union of the B_ℓs. We claim that $v_\ell - v_{\ell-1}$ is *not* in B, *for all* ℓ. This would prove that the sequence v_ℓ is not Cauchy.

Since $B_{\ell'} \cap \mathbb{C}^{n_\ell} \subset B_\ell$, $v_\ell - v_{\ell-1}$ is not in $B_{\ell'}$ for any $\ell' \geq \ell$. And taking the convex hull with $B_{\ell'}$ with $\ell' < \ell$ cannot affect the n_ℓ^{th} entries, since $B_{\ell'} \subset \mathbb{C}^{n_\ell - 1}$ for $\ell' < \ell$. Thus, for every ℓ, $v_\ell - v_{\ell-1}$ is not in the neighborhood B of 0, so the sequence v_ℓ is not Cauchy. ///

[02.5] Show that for complex w the equation $(\frac{d^2}{dx^2} - w^2)\, u_w = 0$ has a solution in $C^2(S^1)$ only when $w \in i\mathbb{Z}$. Let δ^{per} be the $2\pi\mathbb{Z}$ periodic Dirac δ-function, the continuous linear functional on $C^o(\mathbb{R}/2\pi\mathbb{Z})$ given by $\delta^{\mathrm{per}} f = f(0)$. With $w \in \mathbb{C}$ and $w \notin i\mathbb{Z}$, solve

$$\Big(\frac{d^2}{dx^2} - w^2\Big)\, u_w \;=\; \delta^{\mathrm{per}}$$

for u_w on $\mathbb{R}/2\pi\mathbb{Z}$. (*Hint:* expand δ^{per} in a Fourier series.) Identify the residues of the $L^2(S^1)$-valued meromorphic function $w \to u_w$.

The differential equation $(\frac{d^2}{dx^2} - w^2)\, u_w = 0$ on $\mathbb{R}$ has solutions $u_w(x) = Ae^{wx} + Be^{-wx}$ for arbitrary scalars A, B, for $w \neq 0$, and $A + Bx$ for $w = 0$. (The fact that there are no *other* solutions follows from the Mean Value Theorem, for example.) Such a solution *descends* to the quotient $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ only when it is invariant under translation by $2\pi\mathbb{Z}$, which requires $w \in i\mathbb{Z}$, and when $w = 0$ the solution $u(x) = x$ is excluded.

The Fourier series for periodic δ converges in $H^{-\frac{1}{2}-\varepsilon}(S^1)$ for every $\varepsilon > 0$. The equation

$$\sum_n 1 \cdot e^{inx} \;=\; \delta^{\mathrm{per}} \;=\; \Big(\frac{d^2}{dx^2} - w^2\Big)\Big(\sum_n \widehat{u}(n) \cdot e^{inx}\Big) \;=\; \sum_n (-n^2 - w^2) \cdot \widehat{u}(n) \cdot e^{inx} \qquad (\text{in } H^{-1}(S^1))$$

with differentiation extended by continuity to a continuous linear map on Levi-Sobolev spaces, gives $\widehat{u}(n) = -1/(n^2 + w^2)$. For w bounded away from $i\mathbb{Z}$ the series

$$u_w \;=\; -\sum_n \frac{1}{n^2 + w^2} \cdot e^{inx}$$

is convergent in $H^{\frac{3}{2}-\varepsilon}(S^1)$ for all $\varepsilon > 0$, uniformly on compacts. We claim that this is a complex-differentiable $H^{\frac{3}{2}-\varepsilon}(S^1)$-valued function. Estimate the $H^{\frac{3}{2}-\varepsilon}(S^1)$-norm of the difference between the natural difference quotient and the natural expression for the derivative:

$$\Big|\frac{u_{w+h}(x)-u_w(x)}{h} - \sum_n \frac{-2w}{(n^2+w^2)^2}\cdot e^{inx}\Big|^2_{H^{\frac{3}{2}-\varepsilon}(S^1)} \;\le\; \sum_n \Big|\frac{\frac{1}{n^2+(w+h)^2}-\frac{1}{n^2+w^2}}{h} + \frac{2w}{(n^2+w^2)^2}\Big|^2\cdot(1+n^2)^{\frac{3}{2}-\varepsilon}$$

$$= \sum_n \Big|\frac{1}{h(n^2+(w+h)^2)} - \frac{1}{h(n^2+w^2)} + \frac{2w}{(n^2+w^2)^2}\Big|^2\cdot(1+n^2)^{\frac{3}{2}-\varepsilon}$$

$$= \sum_n \Big|\frac{-2w-h}{(n^2+(w+h)^2)(n^2+w^2)} + \frac{2w}{(n^2+w^2)^2}\Big|^2\cdot(1+n^2)^{\frac{3}{2}-\varepsilon}$$

$$= h\cdot\sum_n \Big|\frac{n^2-3w^2-2wh}{(n^2+(w+h)^2)(n^2+w^2)^2}\Big|^2\cdot(1+n^2)^{\frac{3}{2}-\varepsilon} \;\to\; 0$$

as $h\to 0$, uniformly for w in compacts away from $i\mathbb{Z}$, proving the complex differentiability. Thus, the only possible poles are at $w\in i\mathbb{Z}$, and are simple except for the double pole at $w=0$.

Anticipating, as we will soon prove, that the elementary Cauchy-Goursat complex function theory applies to vector-valued functions: at $0\neq in\in i\mathbb{Z}$ there are exactly two terms in the Fourier series with a pole, namely $e^{\pm inx}/(n^2+w^2)$, with residues $e^{\pm inx}/(\pm 2in)$. At 0, the leading term of $1/w^2$ gives the constant 1, and the residue is 0.

Charmingly, apart from the double pole at $w=0$, the residues of u_w produce eigenfunctions! ///

[02.6] Show that there are *no eigenvectors* for the *Volterra operator* $T : L^2[0,1]\to L^2[0,1]$ given by

$$Tf(x) \;=\; \int_0^x f(y)\,dy$$

By design, $\frac{d}{dx}Tf = f$ for $f\in C^o[0,1]$. Show that $(T-\lambda)u = f$ is solvable for u when $0\neq\lambda\in\mathbb{C}$ and $f\in C^1[0,1]$ with $f(0)=0$, and the solution is unique. With T^* the Hilbert-space adjoint of T, show that

$$(TT^*f)(x) \;=\; \int_0^1 \min(x,y)\cdot f(y)\,dy \qquad\qquad (T^*Tf)(x) \;=\; \int_0^1 \big(1-\max(x,y)\big)\cdot f(y)\,dy$$

Find eigenvectors for TT^* and T^*T.

First, a relation $Tf=\lambda\cdot f$ for $f\in L^2$ and $\lambda\neq 0$ would imply f is *continuous*:

$$|\lambda|\cdot|f(x+h)-f(x)| \;=\; |Tf(x+h)-Tf(x)| \;\le\; \int_x^{x+h} 1\cdot|f(t)|\,dt \;=\;\le\; |h|^{\frac{1}{2}}\cdot|f|_{L^2}$$

Then the fundamental theorem of calculus would imply f is continuously differentiable and $\lambda\cdot f' = (Tf)' = f$. Thus, f would be a constant multiple of $e^{x/\lambda}$, by the mean value theorem. However, again from Cauchy-Schwarz-Bunyakowsky, for a λ-eigenfunction

$$|\lambda|\cdot|f(x)| \;\le\; |x|^{\frac{1}{2}}\cdot|f|_{L^2}$$

No non-zero multiple of these exponentials satisfies this equation. Thus, there are no eigenvectors for non-zero eigenvalues.

For $f \in L^2[0,1]$ and $Tf = 0 \in L^2[0,1]$, Tf is almost everywhere 0. In fact, since $x \to Tg(x)$ is unavoidably *continuous*, $Tf(x)$ is 0 for all x. Thus, for all x, y in the interval,

$$0 \;=\; 0-0 \;=\; Tf(y)-Tf(x) \;=\; \int_x^y f(t)\,dt$$

That is, $x \to Tf(x)$ is orthogonal in $L^2[0,1]$ to all characteristic functions of intervals. Finite linear combinations of these are dense in $C^o[0,1]$, which is dense in $L^2[0,1]$. Thus $f = 0$. There are no eigenvectors for the Volterra operator.

One way to solve $(T-\lambda)u = f$ uses the idea that under some circumstances it makes sense to write

$$u \;=\; -\frac{1}{\lambda}\Big(1-\frac{T}{\lambda}\Big)^{-1} f \;=\; -\frac{1}{\lambda}\Big(1+\frac{T}{\lambda}+\frac{T^2}{\lambda^2}+\ldots\Big)f$$

expanding the inverse as a geometric series. Convergence of the infinite sum will follow from estimating the operator norm of T^n. Indeed,

$$T^n f(x) \;=\; \int_0^x\int_0^{x_{n-1}}\ldots\int_0^{x_2}\int_0^{x_1} f(t)\,dt\,dx_1\ldots dx_{n-1} \;=\; \int_0^x f(t)\Big(\int_t^x\int_t^{x_{n-1}}\ldots\int_t^{x_2} dx_1\ldots dx_{n-1}\Big)\,dt$$

$$=\; \int_0^x f(t)\cdot\frac{(x-t)^{n-1}}{(n-1)!}\;dt$$

From this, the operator norm of T^n is at most $1/n!$, so the geometric series expressing $(1-T/\lambda)^{-1}$ converges for all $\lambda \neq 0$. The expression for T^n also gives

$$\Big(1+\frac{T}{\lambda}+\frac{T^2}{\lambda^2}+\ldots\Big)f(x) \;=\; \int_0^x f(t)\cdot\Big(1+\sum_{n\geq 1}\Big(\frac{x-t}{\lambda}\Big)^{n-1}/(n-1)!\Big)\;dt \;=\; f(x)+\int_0^x f(t)\cdot e^{(x-t)/\lambda}\;dt$$

Thus, for $\lambda \neq 0$, we can solve $(T-\lambda)u = f$ by

$$u(x) \;=\; -\frac{f(x)}{\lambda}\int_0^x f(t)\cdot e^{(x-t)/\lambda}\;dt$$

Absence of eigenvectors assures *uniqueness*. For $\lambda = 0$, we need $f(0) = 0$, and then $u = f'$.

The compositions TT^* and T^*T are readily expressed by rewriting the operator T and its adjoint T^* as integrals against *kernel functions* $K(x,y)$, as

$$Tf(x) \;=\; \int_0^1 K(x,y)\,f(y)\;dy \qquad\qquad (\text{where } K(x,y) = \begin{cases} y & (\text{for } y < x) \\ 0 & (\text{for } y > x)\end{cases})$$

The kernel $K^*(x,y)$ of the adjoint is simply $K^*(x,y) = \overline{K(y,x)}$, as a direct computation shows:

$$\int_0^1\Big(\int_0^1 K(x,y)\,f(y)\,dy\Big)\,\overline{g}(x)\;dx \;=\; \int_0^1\int_0^1 K(x,y)\,f(y)\,\overline{g}(x)\;dx\,dy \;=\; \int_0^1 f(y)\;\overline{\int_0^1 K(x,y)\;g(x)\;dx}\,dy$$

Immediately

$$Tf(x) \;=\; \int_0^1 K(x,y)\,f(y)\;dy \qquad\qquad (\text{where } K(x,y) = \begin{cases} 1 & (\text{for } y < x) \\ 0 & (\text{for } y > x)\end{cases})$$

Then

$$K^*(x,y) \;=\; \overline{K(y,x)} \;=\; \begin{cases} 1 & (\text{for } x < y) \\ 0 & (\text{for } x > y)\end{cases} \;=\; \begin{cases} 1 & (\text{for } y > x) \\ 0 & (\text{for } y < x)\end{cases}$$

Thus,

$$TT^*f(x) \;=\; \int_0^1 K(x,y)\int_0^1 K^*(y,t)\,f(t)\;dt\;dy \;=\; \int_0^x\int_y^1 \,f(t)\;dt\;dy$$

$$=\; \int_0^1 f(t)\Big(\int_0^{\min(x,t)} dy\Big)\;dt \;=\; \int_0^1 \min(x,t)\cdot f(t)\;dt$$

A similar computation works for T^*T.

As with T itself, $TT^*f = \lambda\cdot f$ with $\lambda \neq 0$ implies f is C^∞, and $f = -\lambda\cdot f''$, so f is a linear combination of $e^{\pm\xi x}$ where $-\xi^2 = -1/\lambda$. Computing directly:

$$TT^*e^{\xi x} \;=\; \int 0^x y\cdot f(y)\,dy + x\cdot\int_x^1 f(y)\,dy \;=\; \Big[y\cdot\frac{e^{\xi y}}{\xi}\Big]_0^x - \int_0^x \frac{e^{\xi y}\;dy}{\xi} + x\cdot\frac{e^{\xi\cdot 1}-e^{\xi x}}{\xi}$$

$$=\; -\frac{e^{\xi x}}{\xi^2} + \frac{1}{\xi^2} + x\cdot\frac{e^{\xi\cdot 1}}{\xi}$$

The second and third terms are the issue: a linear combination $Ae^{\xi x}+Be^{-\xi x}$ is a $1/\lambda$-eigenvalue if and only if

$$A\cdot\Big(\frac{1}{\xi^2} + x\cdot\frac{e^{\xi\cdot 1}}{\xi}\Big) + B\cdot\Big(\frac{1}{(-\xi)^2} + x\cdot\frac{e^{-\xi\cdot 1}}{-\xi}\Big) \;=\; 0$$

which is

$$A+B=0 \qquad\text{and}\qquad Ae^{\xi}-Be^{-\xi}=0$$

Thus, $B=-A$, and then $e^\xi + e^{-\xi} = 0$, so $\xi \in \pi i\mathbb{Z} + \frac{\pi i}{2}$. That is, for $n\in\mathbb{Z}$, the unique (up to constant multiplies) $(\pi i n + \frac{\pi i}{2})^2$-eigenvector for TT^* is

$$u(x) \;=\; \sin\pi(n+\tfrac{1}{2})x \qquad\qquad \text{(eigenvector for } TT^*)$$

The analogous computation can be done for T^*T, but, alternatively, we can observe that applying T^* to $TT^*f = \lambda\cdot f$ gives $(T^*T)(Tf) = \lambda\cdot(Tf)$. That is, eigenvectors for TT^* are mapped by T to eigenvectors for T^*T. Either way,

$$u(x) \;=\; \cos\pi(n+\tfrac{1}{2})x \qquad\qquad \text{(eigenvector for } T^*T)$$

///

[0.0.1] **Remark:** Since T is *Hilbert-Schmidt*, it is provably *compact*, but $TT^* \neq T^*T$. Thus, we cannot expect T to have eigenvectors, and, indeed, it has none. Nevertheless, TT^* and T^*T *do* have eigenvectors. The same differential equation $f = -\lambda f''$ arises from both $TT^*f = \lambda\cdot f$ and $T^*Tf = \lambda\cdot f$, but, in effect, the implicit boundary conditions differ, so the eigenvectors differ.
