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The incompleteness of weak duals

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The point here is to prove that the weak duals of reasonable topological vector spaces, such as infinite-dimensional Hilbert, Banach, or Fréchet spaces, are *not* complete. That is, in these weak duals there are Cauchy *nets* which *do not converge*.

Happily, this incompleteness is not a fatal problem, because weak duals of Fréchet (and weak duals of strict inductive limits of Fréchet) are *quasi-complete*, and quasi-completeness suffices for most applications.

For example, it is often observed that the space of distributions with the weak topology is *sequentially* complete, but sequential completeness is insufficient for many purposes, such as vector-valued integrals, whether Gelfand-Pettis (weak) or Bochner (strong). Fortunately, spaces of distributions are *quasi-complete.*

It is an exercise to show that in *metrizable* spaces, *sequential* completeness implies completeness.

Weak duals of infinite-dimensional topological vector spaces are rarely metrizable, so the distinctions among *sequentially complete*, *quasi-complete*, and *complete* are meaningful, and potentially significant. Again, *sequential* completeness is insufficient for many applications.

One might worry that quasi-completeness of weak duals is not the best possible result, namely, one might imagine that some of these spaces are *complete*. The discussion here proves that completeness *definitely* does not hold.

Again, fortunately, *quasi*-completeness is sufficient to prove existence of Gelfand-Pettis integrals of continuous compactly-supported vector-valued functions, and to prove that weakly holomorphic vector-valued functions are strongly holomorphic. See [Garrett], for example.

The *incompleteness* (in the sense that not all Cauchy *nets* converge) of most weak duals has been known at least since [Grothendieck 1950], which gives a systematic analysis of completeness of various types of duals. This larger issue is systematically discussed in [Schaefer 1966/99], p. 147-8 and following. The *quasi*-completeness of weak duals and similar spaces is recalled in an appendix here.

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1. *Incompleteness of weak duals of reasonable spaces*

Background material about colimits is recalled in an appendix.

[1.0.1] Theorem: The weak dual of a locally-convex topological vector space V is *complete* if and only if every linear functional on V is *continuous.*

[1.0.2] Corollary: Weak duals of infinite-dimensional Hilbert, Banach, Fréchet, and LF-spaces are definitely *not* complete. ///

[1.0.3] Remark: Nevertheless these duals are *quasi-complete*, which is sufficient for applications.

Proof: An overview of the proof: given a vector space V without a topology, we will topologize V with the finest-possible locally-convex topological vector space topology V_{init}. For this *finest* topology, all linear functionals *are* continuous, and the weak dual *is* complete in the strongest sense. Completeness of the weak dual of any coarser topology on V will be assessed by comparison to V_{init}.

Some set-up is necessary. Given a complex vector space V, let

$$V_{\text{init}} \;=\; \text{locally-convex colimit of finite-dimensional } X \subset V$$

with transition maps being inclusions.

[1.0.4] Proposition: For a locally-convex topological vector space V the identity map $V_{\text{init}} \to V$ is continuous. That is, V_{init} is the finest locally convex topological vector space topology on V.

Proof: Finite-dimensional topological vector spaces have unique topologies. Thus, for any finite-dimensional vector subspace X of V the inclusion $X \to V$ is continuous with that unique topology on X. These inclusions form a compatible family of maps to V, so by the definition of colimit there is a *unique* continuous map $V_{\text{init}} \to V$. This map is the identity on every finite-dimensional subspace, so is the identity on the underlying set V. ///

[1.0.5] Proposition: Every linear functional $\lambda : V_{\text{init}} \to \mathbb{C}$ is *continuous*.

Proof: The restrictions of a given linear function λ on V to finite-dimensional subspaces are compatible with the inclusions among finite-dimensional subspaces. Every linear functional on a finite-dimensional space is continuous, so the defining property of the colimit implies that λ is continuous on V_{init}. ///

[1.0.6] Proposition: The weak dual V^* of a locally-convex topological vector space V injects continuously to the limit of the finite-dimensional Banach spaces

$$V^*_\Phi \;=\; \text{completion of } V^* \text{ under seminorm } p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad (\text{finite } \Phi \subset V)$$

and the weak dual topology is the subspace topology.

Proof: The weak dual topology on the continuous dual V^* of a topological vector space V is given by the seminorms

$$p_v(\lambda) \;=\; |\lambda(v)| \qquad (\text{for } \lambda \in V^* \text{ and } v \in V)$$

Specifically, a local sub-basis at 0 in V^* is given by sets

$$\{\lambda \in V^* \;:\; |\lambda(v)| < \varepsilon\}$$

The corresponding local basis is finite intersections

$$\{\lambda \in V^* \;:\; |\lambda(v)| < \varepsilon, \text{ for all } v \in \Phi\} \qquad (\text{for arbitrary finite sets } \Phi \subset V)$$

These sets contain, and are contained in, sets of the form

$$\{\lambda \in V^* \;:\; \sum_{v\in\Phi} |\lambda(v)| < \varepsilon\} \qquad (\text{for arbitrary finite sets } \Phi \subset V)$$

Therefore, the weak dual topology on V^* is also given by semi-norms

$$p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad (\text{finite } \Phi \subset V)$$

These have the convenient feature that they form a projective family, indexed by (reversed) inclusion. Let $V^*(\Phi)$ be V^* with the p_Φ-topology: this is not Hausdorff, so continuous linear maps $V^* \to V^*(\Phi)$ descend to maps $V^* \to V^*_\Phi$ to the *completion* V^*_Φ of V^* with respect to the pseudo-metric attached to p_Φ. The quotient map $V^*(\Phi) \to V^*_\Phi$ typically has a large kernel, since

$$\dim_{\mathbb{C}} V^*_\Phi \;=\; \text{card}\,\Phi \qquad (\text{for finite } \Phi \subset V)$$

The maps $V^* \to V^*_\Phi$ are compatible with respect to (reverse) inclusion $\Phi \supset Y$, so V^* has a natural induced map to the $\lim_\Phi V^*_\Phi$. Since V separates points in V^*, V^* *injects* to the limit. The weak topology on V^* is exactly the subspace topology from that limit. ///

[1.0.7] Proposition: The weak dual V^*_{init} of V_{init} is the limit of the finite-dimensional Banach spaces

$$V^*_\Phi \;=\; \text{completion of } V^*_{\text{init}} \text{ under seminorm } \; p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad\qquad (\text{finite } \Phi \subset V)$$

Proof: The previous proposition shows that V^*_{init} *injects* to the limit, and that the subspace topology from the limit is the weak dual topology. On the other hand, the limit consists of linear functionals on V, without regard to topology or continuity. Since *all* linear functionals are continuous on V_{init}, the limit is naturally a subspace of V^*_{init}. ///

Returning to the proof of the theorem:

The limit $\lim_\Phi V^*_\Phi$ is a closed subspace of the corresponding *product*, so is *complete* in the strong sense. Any other locally convex topologization V_τ of V has weak dual $V^*_\tau \subset V^*_{\text{init}}$ with the subspace topology, and is *dense* in V^*_{init}. Thus, unless $V^*_\tau = V^*_{\text{init}}$, the weak dual V^*_τ *is not complete.* ///

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2. *Appendix: locally-convex limits and colimits*

We recall some basic but seemingly not-quite-standard facts about (co-) limits of locally convex topological vector spaces.

In the category of locally convex topological vector spaces coproducts (direct sums) exist, namely

$$\coprod_{\alpha\in A} V_\alpha \;=\; \{A-\text{tuples } (\ldots, v_\alpha, \ldots) \text{ where } v_\alpha \in V_\alpha \text{ and } v_\alpha = 0 \text{ for all but finitely-many } \alpha\}$$

with injections of each V_β into the coproduct

$$i_\beta : V_\beta \to \coprod_\alpha V_\alpha \quad \text{by} \quad i_\beta(v_\beta)_\alpha \;=\; \begin{cases} v_\beta & (\text{for } \alpha = \beta) \\ 0 & (\text{otherwise}) \end{cases}$$

The locally convex coproduct topology is the *diamond* topology, with local basis at 0 consisting of sets specified by A-tuples of convex open neighborhoods U_α of 0 in V_α, by

$$U_{\{U_\alpha:\alpha\in A\}} \;=\; \text{convex hull of } \bigcup_{\alpha\in A} i_\alpha(U_\alpha)$$

Colimits in this category are quotients of coproducts:

$$\operatorname{colim} V_\alpha \;=\; \Big(\coprod_\alpha V_\alpha\Big)/\Big(\text{closure of subspace generated by all } i_\alpha(v_\alpha) - i_\beta(\varphi_{\alpha\beta} v_\alpha)\,\Big)$$

where α, β run through all indices with $\alpha \le \beta$, v_α runs over V_α, and where the

$$\varphi_{\alpha\beta} : V_\alpha \to V_\beta$$

are the transition maps specifying the colimit.

The continuous bijection $V_{\text{init}} \to V_\tau$ entails a continuous *injection* $V^*_\tau \to V^*_{\text{init}}$. Since V^*_{init} is a limit of limitands to which V^*_τ surjects (by Hahn-Banach and finite-dimensionality), V^*_τ is dense in V^*_{init}. The weak dual topology on V^*_τ is the subspace topology from V^*_{init}. Since V^*_{init} is *complete*, its dense subspace V^*_τ is complete only if it is equal to V^*_{init}.

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3. *Appendix: ubiquity of quasi-completeness*

This appendix recalls proofs that most interesting topological vector spaces are *quasi-complete*, also recalling relevant definitions.

Lacking a countable local basis at 0, it is necessary to consider *nets* rather than simply *sequences*. Weak duals of infinite-dimensional topological vector spaces are not usually locally countably based.

A **directed set** is a partially ordered set $I, \leq$ such that for $i, j \in I$ there is $k \in I$ such that $i \leq k$ and $j \leq k$. A **Cauchy net** in a topological vector space V is a set $\{s_i : i \in I\} \subset V$ indexed by a directed set I such that, given a neighborhood U of 0 in V, there is $i_o \in I$ such that $s_i - s_j \in U$ for $i, j \geq i_o$.

Similarly, a Cauchy net in a *metric space* X, d is a set $\{s_i : i \in I\} \subset X$ indexed by a directed set I such that, given $\varepsilon > 0$ there is $i_o \in I$ such that $d(s_i, s_j) < \varepsilon$ for $i, j \geq i_o$.

A topological vector space or a metric space is *complete*, in the strongest sense, if every Cauchy net *converges*.

A subset J of a directed set I is **cofinal** if for every $i \in I$ there is $j \in J$ such that $i \leq j$. A typical example of an I with no *countable* cofinal subset is the set of finite subsets of an uncountable set ordered by inclusion.

[3.0.1] Proposition: A *sequentially complete* metric space X, d is *complete*. In particular, Cauchy nets converge, and contain cofinal subsequences converging to the same limit.

Proof: Let $\{s_i : i \in I\}$ be a Cauchy net in X. Given a natural number n, let $i_n \in I$ be an index such that for $i, j \geq i_n$ we have $d(x_i, x_j) < \frac{1}{n}$. Then $\{x_{i_n} : n = 1, 2, \ldots\}$ is a Cauchy sequence. Let x be its limit, using the sequential completeness of X. Given $\varepsilon > 0$, let $j \geq i_n$ be also large enough such that $d(x, x_j) < \varepsilon$. Then

$$d(x, x_{i_n}) \leq d(x, x_j) + d(x_j, x_{i_n}) \;<\; \varepsilon + \frac{1}{n}$$

This is true for every $\varepsilon > 0$, so

$$d(x, x_{i_n}) \;\leq\; \frac{1}{n}$$

We claim that the original Cauchy net also converges to x. Indeed, given $\varepsilon > 0$, take a large-enough natural number n such that $\varepsilon > \frac{1}{n}$. Then for $i \geq i_n$

$$d(x_i, x) \leq d(x_i, x_{i_n}) + d(x_{i_n}, x) \;<\; \varepsilon + \varepsilon$$

with the strict inequality coming from $d(x_{i_n}, x) < \varepsilon$. This proves that the Cauchy subsequence already converges to x. ///

Thus, for metric spaces, and for Hilbert, Banach, and Fréchet spaces, there is no distinction between sequential completeness and completeness in the strongest sense.

A subset B of a topological vector space V is *bounded* if, given a neighborhood U of 0, there is $t_o > 0$ such that $tU \supset B$ for all $t \geq t_o$. This is the general topological vector space version of *boundedness* in metric spaces.

A topological vector space is **quasi-complete** when *bounded* Cauchy nets converge.

Clearly, *closed subspaces* of quasi-complete spaces are quasi-complete. Products (with product topology) and finite sums of quasi-complete spaces are quasi-complete.

[3.0.2] Proposition: A strict colimit of a countable collection of quasi-complete spaces is quasi-complete.

Proof: We saw that bounded subsets of such colimits are exactly the bounded subsets of the limitands. Thus, bounded Cauchy nets in the colimit must be bounded Cauchy nets in one of the closed subspaces. Each of these is assumed quasi-complete, so the colimit is quasi-complete. ///

As a consequence of the proposition, spaces of test functions are quasi-complete, since they are such colimits of the Fréchet spaces of spaces of test functions with prescribed compact support.

Let $\mathrm{Hom}(X, Y)$ be the space of continuous linear functions from a topological vectorspace X to another topological vectorspace Y. Give $\mathrm{Hom}(X, Y)$ the topology induced by the seminorms $p_{x,U}$ where $x \in X$ and U is a convex, balanced neighborhood of 0 in Y, defined by

$$p_{x,U}(T) \;=\; \inf\{t > 0 : Tx \in tU\} \qquad\qquad (\text{for } T \in \mathrm{Hom}(X, Y))$$

[3.0.3] Remark: In the case that X and Y are Hilbert spaces, this construction gives the *strong operator topology* on $\mathrm{Hom}(X, Y)$. Replacing the topology on the Hilbert space Y by its *weak* topology, the construction gives $\mathrm{Hom}(X, Y)$ the *weak operator topology*. That the collection of continuous linear operators is the same in both cases is a consequence of the Banach-Steinhaus theorem, which also plays a role in the following result.

[3.0.4] Theorem: When X is a Fréchet space or LF space, and when Y is quasi-complete, the space $\mathrm{Hom}(X, Y)$, with the topology induced by the seminorms $p_{x,U}$ where $x \in X$ and U is a convex, balanced neighborhood of 0 in Y, is *quasi-complete*.

[3.0.5] Remark: In fact, the starkest hypothesis on $\mathrm{Hom}(X, Y)$ is simply that it support the conclusion of the Banach-Steinhaus theorem. That is, a subset E of $\mathrm{Hom}(X, Y)$ so that the set of all images

$$Ex = \{Tx : T \in E\}$$

is bounded (in Y) for all $x \in X$ is necessarily *equicontinuous*. When X is a Fréchet space, this is true (by the usual Banach-Steinhaus theorem) for *any* Y. Further, by the result above on bounded subsets of special sorts of colimits, we see that the same conclusion holds for X such a colimit.

Proof: Let $E = \{T_i : i \in I\}$ be a bounded Cauchy net in $\mathrm{Hom}(X, Y)$, where I is a directed set. Of course, attempt to define the limit of the net by

$$Tx \;=\; \lim_i T_i x$$

For $x \in X$ the evaluation map $S \to Sx$ from $\mathrm{Hom}(X, Y)$ to Y is continuous. In fact, the topology on $\mathrm{Hom}(X, Y)$ is the coarsest with this property. Therefore, by the quasi-completeness of Y, for each fixed $x \in X$ the net $T_i x$ in Y is bounded and Cauchy, so converges to an element of Y suggestively denoted Tx.

To prove *linearity* of T, fix x_1, x_2 in X, $a, b \in \mathbb{C}$ and fix a neighborhood U_o of 0 in Y. Since T is in the closure of E, for any open neighborhood N of 0 in $\mathrm{Hom}(X, Y)$, there exists

$$T_i \;\in\; E \cap (T + N)$$

In particular, for any neighborhood U of 0 in Y, take

$$N = \{S \in \mathrm{Hom}(X, Y) : S(ax_1 + bx_2) \in U,\; S(x_1) \in U,\; S(x_2) \in U\}$$

Then

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2)$$

$$= \; (T(ax_1 + bx_2) - aT(x_1) - bT(x_2)) - (T_i(ax_1 + bx_2) - aT_i(x_1) - bT_i(x_2))$$

since T_i is linear. The latter expression is

$$T(ax_1 + bx_2) - (ax_1 + bx_2) + a(T(x_1) - T_i(x_1) + b(T(x_2) - T_i(x_2)$$

$$\in \; U + aU + bU$$

By choosing U small enough so that

$$U + aU + bU \;\subset\; U_o$$

we find that

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \;\in\; U_o$$

Since this is true for every neighborhood U_o of 0 in Y,

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \;=\; 0$$

which proves linearity.

To prove *continuity* of the limit operator T, we must first be sure that E is *equicontinuous*. For each $x \in X_j$, the set $\{T_i x : i \in I\}$ is bounded in Y, so by Banach-Steinhaus $\{T_i : i \in I\}$ is an equicontinuous set of linear maps from X_i to Y. (Each X_i is Fréchet.) From the result of the previous section on equicontinuous subsets of LF spaces, E itself is equicontinuous.

Fix a neighborhood U of 0 in Y. Invoking the equicontinuity of E, let N be a small enough neighborhood of 0 in X so that $T(N) \subset U$ for all $T \in E$. Let $x \in N$. Choose an index i sufficiently large so that $Tx - T_i x \in U$, vis the definition of the topology on $\mathrm{Hom}(X, Y)$. Then

$$Tx \in U + T_i x \subset U + U$$

The usual rewriting, replacing U by U' such that $U' + U' \subset U$, shows that T is continuous. ///

[Garrett] P. Garrett, *Notes on functional analysis*, 1990-2011, http://www.math.umn.edu/~garrett/m/fun/

[Grothendieck 1950] A. Grothendieck, *Sur la complétion du dual d'un espace vectoriel localement convexe*, C. R. Acad. Sci. Paris **230** (1950), 605-606.

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