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Plancherel and spectral decomposition/synthesis

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This gives some context for *Plancherel* and *spectral decomposition*. We want to consider Fourier series and Fourier transforms as completely simple, straightforward, and intuitive, as suggestive models for automorphic spectral theory and Sobolev theory.

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1. *Plancherel for discrete spectral decompositions*

The primary example in this section is *Fourier series*, and differential operator $\Delta = d^2/dx^2$. The Plancherel theorem in this particular case is due to Parseval.

[1.1] Elementary, abstract Plancherel Given an orthonormal basis [1] $\{v_i\}$ in a Hilbert space V, by definition any vector $v \in V$ is a limit of finite linear combinations of the $\{v_i\}$. In fact, a stronger, simpler assertion is true: v is expressible as an abstract *Fourier series* [2]

$$v \;=\; \sum_i \langle v, v_i\rangle \cdot v_i \qquad\qquad \text{(convergent in } V\text{)}$$

meaning that v is the limit of the *finite* partial subsums. Proof is just below. In this abstract setting, *Plancherel's (Parseval's) theorem* is relatively elementary, asserting

$$|v|^2 \;\;=\;\; \sum_i |\langle v, v_i\rangle|^2$$

We prove the abstract Fourier expansion and Plancherel theorem together. For fixed n, from the orthogonality of the n^{th} tail $v - \sum_{i\le n}\langle v, v_i\rangle \cdot v_i$ to the basis vectors v_i for $i \le n$, we first obtain *Bessel's inequality*

$$|v|^2 \;=\; \Big|\sum_{i\le n}\langle v, v_i\rangle \cdot v_i\Big|^2 + \Big|v - \sum_{i\le n}\langle v, v_i\rangle \cdot v_i\Big|^2 \;\ge\; \sum_{i\le n} |\langle v, v_i\rangle|^2$$

In particular,

$$\sum_i |\langle v, v_i\rangle|^2 \;\le\; |v|^2$$

so the infinite sum is convergent, implying a trivial case of a Riemann-Lebesgue lemma: $\langle v, v_i\rangle \to 0$.

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[1] Recall the elementary convention that a *basis* $\{v_i\}$ in a Hilbert space is *not* a vector-space basis, but is a basis for a *dense* subspace. Indeed, that dense subspace consists precisely of *finite* linear combinations of the Hilbert-space basis vectors.

[2] Apparently Fourier did not possess the inner-product expression for (concrete) Fourier coefficients when he first proposed that functions are expressible in Fourier series. This understandably weakened his claim. Worse, at the time, notions of *convergence* and even the notion of *function* were amorphous, so that the sense(s) in which a function might be *represented by* a Fourier series could not be discussed easily.

More than that, the sequence of partial sums $s_n = \sum_{i\le n} \langle v, v_i\rangle \cdot v_i$ is *Cauchy* in V, so converges to *some* v'. That $v' = v$ follows from *continuity* of $\langle,\rangle$ and

$$\langle v - v', v_j\rangle \;=\; \langle v, v_j\rangle - \langle \lim_n s_n,\, v_j\rangle \;=\; \langle v, v_j\rangle - \langle v, v_j\rangle \;=\; 0$$

Since $\{v_j\}$ is a Hilbert-space basis, $\lim_n s_n = v$, proving Plancherel. ///

[1.2] Eigenvector/spectral expansions

If those $\{v_i\}$ were *eigenvectors* for a *continuous* linear operator T on V, with eigenvalues $Tv_i = \lambda_i v_i$, then the abstract Fourier expansion of $v \in V$ as an infinite linear combination $v = \sum_i c_i\, v_i$ allows easy computation:

$$Tv \;=\; T\Big(\sum_i c_i\, v_i\Big) \;=\; \sum_i c_i\, Tv_i \;=\; \sum_i c_i\, \lambda_i\, v_i$$

Note that T passes inside the infinite sum because T is *continuous*. Continuous operators T are *bounded*, so the eigenvalues λ_i are *bounded* in absolute value, and the eigenvector expansion for Tv still converges in V.

A Hilbert-space orthonormal basis of eigenvectors exists for *self-adjoint compact* operators[3] T, for example. This is the content of the *spectral theorem* for such operators.

For any more general class of operators, there need not be an orthonormal Hilbert-space basis of eigenfunctions. Yet, in important examples, more can be done, as we do a little later for Fourier *transforms*.

[1.3] Example of elementary, abstract Plancherel

Granting that the exponentials $\psi_\xi(x) = e^{2\pi i x\xi}$ with $\xi \in \mathbb{Z}$ form an orthonormal basis for $L^2[0,1]$, we first have a *Fourier expansion*

$$f \;=\; \sum_{\xi\in\mathbb{Z}} \langle f, \psi_\xi\rangle \cdot \psi_\xi \qquad\qquad (\text{convergent in } L^2[0,1])$$

This says nothing about *pointwise* convergence, and we do not *expect* it to. The abstract elementary Plancherel applied to this example asserts that

$$|f|^2 \;=\; \sum_{\xi\in\mathbb{Z}} |\langle f, \psi_\xi\rangle|^2$$

Since the exponentials are eigenfunctions for the ubiquitous operator $\Delta = d^2/dx^2$, say that a Fourier expansion is a *spectral* expansion, or *eigenfunction* expansion. The precise nature of convergence must be clarified in context, as below.

The exponentials are also simultaneous eigenvectors/eigenfunctions for the *translation* operators $R_y f(x) = f(x+y)$. In fact, d/dx is the *infinitesimal* version of translation, in the sense that the fundamental theorem of calculus gives various useful relations:

$$\int_0^x f'(t)\,dt \;=\; f(x) - f(0) \qquad\qquad \text{and/or} \qquad\qquad f(x) + \int_x^{x+y} f'(t)\,dt \;=\; f(x+y)$$

[1.4] Unbounded operators

The ubiquitous linear operator $T = \Delta = d^2/dx^2$ is not a continuous linear endomorphism of $L^2[0,1]$. This is not a matter of the existence of non-differentiable functions in $L^2[0,1]$. Rather, even on differentiable functions such as the exponentials ψ_ξ,

$$\frac{|\frac{d^2}{dx^2}\psi_\xi|}{|\psi_\xi|} \;=\; \frac{4\pi^2|\xi|^2}{1} \;\longrightarrow\; +\infty \qquad\qquad (\text{as } |\xi| \to \infty)$$

[3] Recall that a compact operator on a Hilbert space is an operator-norm limit of finite-rank operators.

That is, on the (dense) subspace of differentiable functions in $L^2[0,1]$, T is not *bounded*, so cannot be *continuous*. It is *not* a matter of artificially defining an operator on non-differentiable functions.

[1.5] Eigenvectors for unbounded operators Although $\Delta = d^2/dx^2$ is in no way a continuous operator on $V = L^2[0,1]$, obviously the exponentials ψ_ξ are eigenvectors. More precisely, Δ is a well-defined linear operator stabilizing the subspace V_o of V consisting of *smooth* functions on $[0,1]$. This legitimizes speaking of the ψ_ξ as *eigenfunctions* or *eigenvectors* of Δ.

We are lucky here, that the troublesomely unbounded, but natural, operator Δ has eigenvectors in V giving an orthonormal basis for V. In general, there is no reason to expect this.

Here, as generally, proof that an unbounded operator nevertheless has enough eigenvectors to make up an orthonormal basis amounts to making sense of, and proving, that the *resolvent* $R_z = (\Delta - z^2)^{-1}$ with $z \in \mathbb{C}$ is a *compact operator* away from its poles. [4] Of course, most unbounded operators do not have compact resolvents.

[1.6] Compact resolvents The principal device to prove that an operator $(\Delta - z^2)^{-1}$ is *compact*, if it is so, is to express it as a nice-enough integral operator to demonstrate that it is *Hilbert-Schmidt*. [5] We want a function $K(x,y) = K_z(x,y)$ of two variables, preferably continuous, solving $(\Delta - z^2)u = f$ by

$$u(x) \;=\; \int_0^1 K(x,y)\, f(y)\, dy$$

Applying $\Delta - z^2$ to this and assuming [6] we can pass the differential operator inside the integral,

$$f(x) \;=\; (\Delta - z^2)u(x) \;=\; \int_0^1 (\Delta_x - z^2)K(x,y)\, f(y)\, dy$$

Thus, we want $(\Delta_x - z^2)K(x,y) = \delta_x(y)$, with Dirac delta. [7] The standard procedure [8] to obtain $K(x,y)$ is as follows. Take solutions α, β to the *homogeneous* equation $(\Delta - z^2)u = 0$, such that $\alpha(0) = 0$ and $\beta(1) = 0$. In fact, the solutions to $(\Delta - z^2)u = 0$ are $e^{\pm zx}$, so we can take

$$\alpha(x) \;=\; \sinh zx \qquad \text{and} \qquad \beta(x) \;=\; \sinh z(x-1)$$

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[4] Using $\Delta - z^2$ instead of $\Delta - z$ avoids square roots subsequently.

[5] Recall that a *Hilbert-Schmidt* operator $T : L^2(X) \to L^2(Y)$ is one given by a kernel $K(x,y) \in L^2(X \times Y)$, by $Tf(y) = \int_X K(x,y)\, f(x)\, dx$. It is standard, and not difficult, that these are compact operators.

[6] Differentiating in a parameter under the integral sign is justified by the Gelfand-Pettis theory of so-called *weak integrals*. This is not completely elementary, but is straightforward.

[7] In this one-variable situation, an intuitive or heuristic view of δ_x suffices to suggest the correct course. For example, $\delta_x(y)$ is the derivative (in y) of the step function

$$H_x(y) \;=\; \begin{cases} 0 & (\text{for } y < x) \\ 1 & (\text{for } y > x) \end{cases}$$

That is, δ_x is a *spike* of total mass 1 concentrated at x.

[8] The kernel K has many classical names, for example *Green's function*. The procedure here applies to exhibit compact resolvents of somewhat more general second-order one-variable differential equations, called *Sturm-Liouville* problems. Kodaira and Titchmarsh analyzed general ordinary differential equations, on *infinite* intervals or with other weakened hypotheses, and showed that the resolvents are *not* necessarily compact in that generality.

We want to define $K(x,y)$ to be a multiple of $\alpha(x)$ for $x<y$ and a multiple of $\beta(x)$ for $x>y$, so that

$$\Delta_x K(x,y) \;=\; \delta_x(y)$$

That is, let

$$K(x,y) \;=\; \begin{cases} a(y)\cdot\alpha(x) & (\text{for } x<y) \\ \\ b(y)\cdot\beta(x) & (\text{for } x>y) \end{cases}$$

The requirement $\Delta_x K(x,y) = \delta_x$ is that the two curve fragments match at $x=y$, and that the slope increases by 1 moving from one fragment to another:

$$a(y)\,\alpha(y) \;=\; b(y)\,\beta(y) \qquad \text{and} \qquad a(y)\,\alpha'(y)+1 \;=\; b(y)\,\beta'(y) \qquad\qquad (\text{for all } y)$$

These two linear equations in two unknowns $a(y), b(y)$ are readily solved: from the first,
$b(y) = a(y)\alpha(y)/\beta(y)$, and substituting into the second gives

$$a\alpha'+1 \;=\; \frac{a\alpha}{\beta}\,\beta'$$

so

$$a\Big(\alpha' - \frac{\alpha\beta'}{\beta}\Big) \;=\; -1$$

and

$$a \;=\; \frac{-1}{\alpha' - \frac{\alpha\beta'}{\beta}} \;=\; \frac{-\beta}{\alpha'\beta-\alpha\beta'} \qquad \text{and} \qquad b \;=\; \frac{a\,\alpha}{\beta} \;=\; \frac{-\alpha}{\alpha'\beta-\alpha\beta'}$$

The denominator is the *Wronskian* $W(\alpha,\beta)$. Since

$$\frac{d}{dy}W(\alpha,\beta) \;=\; (\alpha'\beta)'-(\alpha\beta')' \;=\; \alpha''\beta+\alpha'\beta'-\alpha'\beta'-\alpha\beta'' \;=\; (z^2\alpha)\beta-\alpha(z^2\cdot\beta) \;=\; 0$$

the Wronskian is *constant*, and it suffices to evaluate it at $y=0$:

$$W(\alpha,\beta)(0) \;=\; \alpha'(0)\cdot\beta(0)-\alpha(0)\cdot\beta'(0) \;=\; \alpha'(0)\cdot\beta(0)-0\cdot\beta'(0) \;=\; \alpha'(0)\cdot\beta(0)$$

$$=\; z\cosh(z\cdot 0)\cdot\sinh z(0-1) \;=\; -z\sinh z$$

Thus, the Wronskian is non-vanishing for $z\notin\pi i\mathbb{Z}$, and

$$\begin{aligned} a(y) \;&=\; \frac{-\beta}{-z\sinh z} \;=\; \frac{-\sinh z(y-1)}{-z\sinh z} \;=\; \frac{\sinh z(y-1)}{z\sinh z} \\ b(y) \;&=\; \frac{-\alpha}{-z\sinh z} \;=\; \frac{-\sinh zy}{-z\sinh z} \;=\; \frac{\sinh zy}{z\sinh z} \end{aligned}$$

Then

$$K(x,y) \;=\; \begin{cases} a(y)\alpha(x) & (\text{for } x<y) \\ \\ b(y)\beta(x) & (\text{for } x>y) \end{cases} \;=\; \begin{cases} \frac{-\beta(y)}{W(\alpha,\beta)}\alpha(x) & (\text{for } x<y) \\ \\ \frac{-\alpha(y)}{W(\alpha,\beta)}\beta(x) & (\text{for } x>y) \end{cases}$$

$$=\; \begin{cases} \dfrac{\sinh z(y-1)\cdot\sinh zx}{z\sinh z} & (\text{for } x<y) \\ \dfrac{\sinh zy\cdot\sinh z(x-1)}{z\sinh z} & (\text{for } x>y) \end{cases}$$

The point is that for $z\notin\pi i\mathbb{Z}$ this is continuous on $[0,1]\times[0,1]$, so certainly L^2 there. Thus, $K(x,y)$ gives a Hilbert-Schmidt operator, which is *compact*. That is, Δ has compact resolvent on $L^2[0,1]$.

Indeed, the essential qualitative point $K(x,y) \in C^o([0,1]\times[0,1])$ does not need many of the details above.

[1.7] Δ-eigenvectors from resolvent eigenvectors The visible symmetry $K(x,y) = K(y,x)$ implies that $K(x,y)$ gives a *self-adjoint* compact operator for $K(x,y)$ real-valued. Thus, for $0 < z \in \mathbb{R}$, the operator attached to each kernel $K(x,y) = K_z(x,y)$ with $0 < z$ is compact and self-adjoint, so $L^2[0,1]$ has an orthonormal Hilbert-space basis of eigenvectors for each such operator.

Recall Hilbert's argument that resolvents *commute*: for $\lambda,\mu \in \mathbb{C}$

$$(T-\lambda)^{-1} - (T-\mu)^{-1} \;=\; (T-\lambda)^{-1}\big[(T-\mu)-(T-\lambda)\big](T-\mu)^{-1} \;=\; (T-\lambda)^{-1}(\lambda-\mu)(T-\mu)^{-1}$$

Also,

$$(T-\lambda)^{-1} - (T-\mu)^{-1} \;=\; (T-\mu)^{-1}\big[(T-\mu)-(T-\lambda)\big](T-\lambda)^{-1} \;=\; (T-\mu)^{-1}(\lambda-\mu)(T-\lambda)^{-1}$$

However, what we really want is an assertion that eigenvectors for any one of the resolvents $K_z(x,y)$ are eigenvectors for $T = \Delta$. To this end, we would like the *domain* of T to include the *images* of the resolvents.

This would require *closed-ness* of the properly-defined T, in the sense that its *graph* is a closed subset of $L^2[0,1]\times L^2[0,1]$. Unsurprisingly, taking Δ to have domain consisting of *smooth* functions in $L^2[0,1]$ is too restrictive: we must take the (graph) *closure* T of Δ.

Applying $(T-\lambda)$ to $(T-\lambda)^{-1}v = \mu\cdot v$ with $\lambda,\mu\in\mathbb{C}$ gives

$$v \;=\; (T-\lambda)(T-\lambda)^{-1}v = (T-\lambda)(\mu\cdot v) \;=\; \mu\cdot(T-\lambda)v \;=\; \mu Tv - \mu\lambda v$$

which gives the expected outcome

$$Tv \;=\; \Big(\frac{1}{\mu}+\lambda\Big)\cdot v$$

That is, eigenvectors for the resolvents are eigenvectors for (the closure of) the unbounded operator Δ. [9]

[1.8] Solving differential equations via spectral expansions Viewing Fourier series of $L^2[0,1]$ functions as expansions in terms of Δ-eigenvectors facilitates solution of differential equations. In effect, Fourier series *diagonalize* the differential operator Δ.

For example, to solve $(\Delta - z^2)u = f$ for u, write u and f in their Fourier expansions $u = \sum_\xi \widehat{u}(\xi)\,\psi_\xi$ and $f = \sum_\xi \widehat{f}(\xi)\,\psi_\xi$. The natural impulse is to attempt to differentiate *termwise* in the sum

$$u(x) \;=\; \sum_\xi \widehat{u}(\xi)\,\psi_\xi(x) \qquad\qquad \text{(equality in an } L^2 \text{ sense)}$$

Postponing justification of termwise differentiation to the discussion of Sobolev spaces just below,

$$(\Delta_x - z^2)u(x) \;=\; (\Delta_x - z^2)\sum_\xi \widehat{u}(\xi)\,\psi_\xi(x) \;=\; \sum_\xi \widehat{u}(\xi)\,(\Delta_x - z^2)\psi_\xi(x) \;=\; \sum_\xi \widehat{u}(\xi)\,(-4\pi^2\xi^2 - z^2)\psi_\xi(x)$$

Fourier expansions are unique because the exponentials are an orthonormal basis, so

$$\widehat{f}(\xi) \;=\; (-4\pi^2\xi^2 - z^2)\,\widehat{u}(\xi)$$

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[9] In fact, Δ is *elliptic*, so its eigenvectors *are* smooth, by *elliptic regularity*. However, compactness of the resolvent is a stronger condition, assuring existence of a basis of eigenvectors. Not all elliptic operators have eigenvectors, as of $\Delta = d^2/dx^2$ on $L^2(\mathbb{R})$ illustrates.

and $\widehat{u}(\xi) = \widehat{f}(\xi)/(-4\pi^2\xi^2 - z^2)$, so

$$u \;=\; \sum_\xi \frac{\widehat{f}(\xi)}{-4\pi^2\xi^2 - z^2}\,\psi_\xi \qquad\qquad \text{(equality in an } L^2 \text{ sense???)}$$

The indicated division improves L^2 convergence of the Fourier series, so $f \in L^2[0,1]$ gives $u \in L^2[0,1]$. Interchange of differentiation and summation, and related convergence questions, are treated in the following subsection.

[1.9] Introduction to Sobolev spaces on $\mathbb{R}/Z$

Since Fourier series are eigenfunction expansions for Δ, we would *imagine* that for $u = \sum_\xi c_\xi\,\psi_\xi$ in $L^2[0,1]$,

$$\Delta u \;=\; \Delta\Big(\sum_\xi c_\xi\,\psi_\xi\Big) \;=\; \sum_\xi (-4\pi^2\xi^2)\cdot c_\xi\,\psi_\xi$$

However, convergence has been weakened, and the image cannot be in $L^2[0,1]$ for general $u \in L^2[0,1]$.

Nevertheless, the resulting Fourier expansion has meaning, via Plancherel. It is not obvious, but these questions motivate considering functions on $[0,1]$ as $\mathbb{Z}$-periodic functions on $\mathbb{R}$, or, equivalently, functions on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Thus, rather than $C^\infty[0,1]$ without comparison conditions on the endpoints, we consider $C^\infty(S^1)$, smooth functions on $[0,1]$ whose values and derivatives' values *match* at the endpoints.

With this notion of smoothness, *integration by parts* leaves no boundary terms, and

$$\Big(\frac{du}{dx}\Big)\widehat{\ }(\xi) \;=\; \int_0^1 e^{-2\pi i\xi x}\,u'(x)\,dx \;=\; -\int_0^1 \frac{d}{dx}e^{-2\pi i\xi x}\,u(x)\,dx \;=\; 2\pi i\xi\int_0^1 e^{-2\pi i\xi x}\,u(x)\,dx \;=\; 2\pi i\xi\,\widehat{u}(\xi)$$

Since all derivatives of smooth u are in $L^2[0,1]$, by Riemann-Lebesgue applied to the derivatives, $|\xi|^n\cdot\widehat{u}(\xi) \to 0$ for all n. That is, Fourier coefficients of smooth functions (with endpoint matching) decrease rapidly, in the sense of decreasing faster than any $1/|\xi|^n$.

In terms of decrease of Fourier coefficients, for $s \geq 0$ the s^{th} Sobolev space is

$$H^s(S^1) \;=\; \{u \in L^2[0,1] : \sum_\xi (1+|\xi|^2)^s\cdot|\widehat{u}(\xi)|^2 < \infty\}$$

It is immediate that $H^s(S^1)$ is the completion of $C^\infty(S^1)$ with respect to the s^{th} Sobolev norm $|*|_s$ given by

$$|u|_s^2 \;=\; \sum (1+|\xi|^2)^s\cdot|\widehat{u}(\xi)|^2$$

since the finite subsums are smooth.

Then, *by design* $\Delta : C^\infty(S^1) \to C^\infty(S^1)$ is *continuous* when the source is given the $H^s(S^1)$ topology and the target is given the $H^{s-2}(S^1)$ topology. Extend *by continuity* to a continuous map $\Delta : H^s(S^1) \to H^{s-2}(S^1)$.

The normalization of the indexing scheme is remembered by observing that the degree-2 operator Δ (or its continuous extension) should map $H^s(S^1)$ to $H^{s-2}(S^1)$.

In this context, Fourier series with coefficients of polynomial growth, such as those which might arise by applying Δ to not-strongly-convergent Fourier series, have a natural meaning as continuous linear functionals (maps to $\mathbb{C}$) on positively-indexed Sobolev spaces, as follows.

Specifically, define negatively-indexed Sobolev spaces $H^{-s}(S^1)$ as spaces of *distributions*[10]

$$H^{-s}(S^1) \;=\; \{\text{distributions}\, v \;:\; \sum_\xi (1+\xi^2)^{-s}\,|v(\psi_\xi)|^2 \;<\; \infty\}$$

Write $\widehat{v}(\xi)$ for $\widehat{v}(\psi_\xi)$. For $u \in H^s(\mathbb{R})$ and $v \in H^{-s}(\mathbb{R})$ with $s \geq 0$, there is the natural $\mathbb{C}$-bilinear pairing

$$\langle u, v\rangle_{H^s \times H^{-s}} \;=\; \sum_\xi \widehat{u}(\xi)\cdot\widehat{v}(\xi)$$

A weighted version of Cauchy-Schwarz-Bunyakowsky proves convergence (and continuity of the pairing):

$$|\langle u, v\rangle_{H^s\times H^{-s}}| \;=\; \Big|\sum_\xi \widehat{u}(\xi)\cdot\widehat{v}(\xi)\Big| \;=\; \Big|\sum_\xi \Big(\widehat{u}(\xi)\cdot(1+|\xi|^2)^{s/2}\Big)\cdot\Big((1+|\xi|^2)^{-s/2}\cdot\widehat{v}(\xi)\Big)\Big|$$

$$\leq\; \sum_\xi (1+|\xi|^2)^s\cdot|\widehat{u}(\xi)|^2 \;\;\cdot\;\; \sum_\xi (1+|\xi|^2)^{-s}\cdot|\widehat{v}(\xi)|^2 \;\;=\;\; |u|_s^2\cdot|v|_{-s}^2$$

That is, Fourier series not in L^2 give continuous linear functionals on subspaces of $L^2[0,1]$ consisting of Fourier series converging more strongly.

The above pairing is obviously an extension of the Plancherel pairing of $L^2[0,1] = H^0(S^1)$ with itself.

[1.9.1] Remark: Thus, termwise differentiation is always justified, *if* interpreted as L^2-differentiation, and the resulting series converging in an appropriate Sobolev space. Connection to classical differentiability and continuity are best made slightly indirectly, as follows.

[1.10] Sobolev's lemma/imbedding

It is natural to wonder whether functions u with sufficiently many *distributional* derivatives in $L^2[0,1]$ are differentiable in a classical sense.

Indeed, the differentiation $\Delta : H^s(S^1) \to H^{s-2}(S^1)$ is an *extension* from the dense subspace $C^\infty(S^1)$, so the relation to classical differentiability is *not* tautological. [11]

The natural Banach-space structure on $C^k(S^1)$ is given by [12]

$$|f|_{C^k} \;=\; \sup_{0\leq i\leq k}\,\sup_{x\in S^1} |f^{(i)}(x)|$$

The claim is that, for $u \in C^\infty(S^1)$,

$$|f|_{C^k} \;\ll_{s,k}\; |f|_{H^s} \qquad\qquad (\text{for all } s > k + \tfrac{1}{2})$$

That is, H^s-limits of Fourier series also converge in C^k. We prove this for $k = 0$. That is, we show that $H^s(S^1) \subset C^o(S^1)$ for all $s > \frac{1}{2}$.

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[10] One definition of *distributions* on S^1 is as the completion of $C^\infty(S^1)$ with respect to the weak *-topology, the latter given by $\nu_u(v) = \int_{S_1} u(x)\,v(x)\,dx$ for $u \in C^\infty(S^1)$. Equivalently, the space of distributions is the space of continuous linear functionals on $C^\infty(S^1)$, with weak *-topology given by the same semi-norms, perhaps written $\nu_u(v) = v(u)$. Various argument prove density of smooth functions in distributions (in the weak *-topology), which proves that these two characterizations refer to the same things.

[11] When necessary for clarity, the differentiation $\Delta : H^s(S^1) \to H^{s-2}(S^1)$ obtained by *extension by continuity* in those topologies is called L^2-differentiation. It is compatible with *distributional* differentiation, but tracks different topologies. Especially, the topology on the target $H^{s-2}(S^1)$ is much finer than the weak $*$-topology on distributions.

[12] There are some things to be checked here, to be sure that all the plausible features are genuine.

The argument is by the same weighted version of Cauchy-Schwarz-Bunyakowsky as above:

$$\Big|\sup_x \Big(\sum_\xi c_\xi\,\psi_\xi(x)\Big)\Big| \;\le\; \sup_x \sum_\xi |c_\xi| \;=\; \sum_\xi |c_\xi| \;=\; \sum_\xi \Big(|c_\xi|\cdot(1+|\xi|^2)^{s/2}\Big)\cdot\frac{1}{(1+|\xi|^2)^{s/2}}$$

$$\le\; \sum_\xi |c_\xi|^2\cdot(1+|\xi|^2)^s \;\;\cdot\;\; \sum_\xi \frac{1}{(1+|\xi|^2)^s} \;\;=\;\; \Big|\sum_\xi c_\xi\,\psi_\xi\Big|^2_{H^s} \;\;\cdot\;\; \sum_\xi \frac{1}{(1+|\xi|^2)^s}$$

For $s > \frac{1}{2}$ the latter sum is convergent. Thus, suitable Sobolev norms dominate classical C^k norms related to continuity and pointwise differentiability. This proves Sobolev's inequality, also called Sobolev's *imbedding*, for functions on S^1, that is, for periodic functions on $\mathbb{R}$.

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2. *Plancherel for continuous spectral decompositions*

The primary example in this section is *Fourier transform*, and differential operator $\Delta = d^2/dx^2$.

[2.1] Generalized eigenvectors Except for the happy cases of compact self-adjoint operators or operators with compact resolvents, there is no reason to expect a basis of eigenvectors for a given operator T, continuous or not.

For self-adjoint operators, there are standard *spectral theorems* decomposing the Hilbert space via *projection-valued measures*. This and other general, abstract decompositions are reassuring, but insufficient.

The first important natural example of eigenvectors falling outside the given Hilbert space is in the action of $T = \Delta$ on[13] $L^2(\mathbb{R})$. The differential equation $u'' - z^2 u = 0$ has solutions $u(x) = e^{\pm zx}$. None of these is in $L^2(\mathbb{R})$. That is, Δ has *no* eigenvectors in $L^2(\mathbb{R})$. Nevertheless, Fourier inversion gives a tangible eigenvector decomposition, as follows.

[2.2] Plancherel for Fourier transform, generalized eigenvectors Recall the *Fourier transform*

$$\widehat{f}(\xi) \;=\; \mathscr{F}f(\xi) \;=\; \int_{\mathbb{R}} \overline{\psi}_\xi(x)\,f(x)\,dx \qquad\qquad (\text{with } \psi_\xi(x) = e^{2\pi i\xi x})$$

These integrals converge nicely for f in the space $\mathscr{S}(\mathbb{R})$ of *Schwartz*[14] functions. When we know how to justify[15] moving the differentiation under the integral,

$$\frac{d}{d\xi}\widehat{f}(\xi) \;=\; \frac{d}{d\xi}\int_{\mathbb{R}} \overline{\psi}_\xi(x)\,f(x)\,dx \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial\xi}\overline{\psi}_\xi(x)\,f(x)\,dx$$

$$=\; \int_{\mathbb{R}} (-2\pi i x)\,\overline{\psi}_\xi(x)\,f(x)\,dx \;=\; (-2\pi i)\int_{\mathbb{R}} \overline{\psi}_\xi(x)\,xf(x)\,dx \;=\; (-2\pi i)\widehat{xf}(\xi)$$

Similarly, with an integration by parts,

$$-2\pi i\xi\cdot\widehat{f}(\xi) \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial x}\overline{\psi}_\xi(x)\cdot f(x)\,dx \;=\; -\mathscr{F}\frac{df}{dx}(\xi)$$

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[13] Although Δ cannot be reasonably defined on all of $L^2(\mathbb{R})$, the *style* is to say that it acts *on* $L^2(\mathbb{R})$.

[14] As usual, the space of Schwartz functions consists of infinitely-differentiable functions all of whose derivatives are of *rapid decay*, that is, decay more rapidly at $\pm\infty$ than every $1/|x|^N$.

[15] Interchange of this integration and differentiation is best understood via Gelfand-Pettis integrals.

Then it is immediate that $\mathscr{F}$ maps $\mathscr{S}(\mathbb{R})$ to itself. The *Plancherel theorem*

$$|\mathscr{F}f| \;=\; |f| \qquad\qquad (L^2(\mathbb{R}) \text{ norm, for } f \in \mathscr{S}(\mathbb{R}))$$

extends the Fourier transform to $L^2(\mathbb{R})$ *by continuity*, despite the general divergence of the literal integral.

[2.3] Fourier inversion Schwartz functions on $\mathbb{R}$ are *superpositions* of exponentials ψ_ξ with $\xi \in \mathbb{R}$, by *Fourier inversion*:

$$f(x) \;=\; \int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi$$

The exponentials *are* eigenfunctions for Δ, but are not in $L^2(\mathbb{R})$. Thus, these exponentials are appropriate *generalized* eigenfunctions for Δ on $L^2(\mathbb{R})$, because, although not in $L^2(\mathbb{R})$, there *are* no Δ-eigenfunctions in $L^2(\mathbb{R})$, and elements of $L^2(\mathbb{R})$ are superpositions of these eigenfunctions ψ_ξ. Since the superposition is an *integral*[16] this spectral decomposition is called *continuous*.

Fourier inversion is proven first for $\mathscr{S}(\mathbb{R})$. The natural idea, that unfortunately begs the question, is the obvious:

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; \int_{\mathbb{R}} \psi_\xi(x) \Big(\int_{\mathbb{R}} \overline{\psi}_\xi(t)\, f(t)\, dt \Big)\, d\xi \;=\; \int_{\mathbb{R}} f(t) \Big(\int_{\mathbb{R}} \psi_\xi(x-t)\, dt \Big)\, dt$$

If we could *justify* asserting that the inner integral is $\delta_x(t)$, which it *is*, then Fourier inversion follows. However, Fourier inversion for $\mathscr{S}(\mathbb{R})$ is used to make sense of that inner integral in the first place.

Despite the impasse, the situation is encouraging. A dummy *convergence factor* will legitimize the idea. For example, let $g(x) = e^{-\pi x^2}$ be the usual Gaussian. Various computations show that it is its own Fourier transform. For $\varepsilon > 0$, as $\varepsilon \to 0^+$, the dilated Gaussian $g_\varepsilon(x) = g(\varepsilon \cdot x)$ approaches 1 uniformly on compacts. Thus,

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; \int_{\mathbb{R}} \lim_{\varepsilon \to 0^+} g(\varepsilon\xi)\; \psi_\xi(x)\; \widehat{f}(\xi)\, d\xi \;=\; \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} g(\varepsilon\xi)\; \psi_\xi(x)\; \widehat{f}(\xi)\, d\xi$$

by *monotone convergence* or more elementary reasons. Then the iterated integral is legitimately rearranged:

$$\int_{\mathbb{R}} g(\varepsilon\xi)\; \psi_\xi(x)\; \widehat{f}(\xi)\, d\xi \;=\; \int_{\mathbb{R}} \int_{\mathbb{R}} g(\varepsilon\xi)\; \psi_\xi(x)\; \overline{\psi}_\xi(t)\, f(t)\, dt\, d\xi \;=\; \int_{\mathbb{R}} \int_{\mathbb{R}} g(\varepsilon\xi)\; \psi_\xi(x-t)\, f(t)\, d\xi\, dt$$

By changing variables in the definition of Fourier transform, $\widehat{g_\varepsilon} = \frac{1}{\varepsilon} g_{1/\varepsilon}$. Thus,

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; \int_{\mathbb{R}} \frac{1}{\varepsilon}\, g\Big(\frac{x-t}{\varepsilon}\Big)\, f(t)\, dt \;=\; \int_{\mathbb{R}} \frac{1}{\varepsilon}\, g\Big(\frac{t}{\varepsilon}\Big) \cdot f(x+t)\, dt$$

The sequence of function $g_{1/\varepsilon}/\varepsilon$ is not an *approximate identity* in the strictest sense, since the supports are the entire line. Nevertheless, the integral of each is 1, and as $\varepsilon \to 0^+$, the mass is concentrated on smaller and smaller neighborhoods of $0 \in \mathbb{R}$. Thus, for $f \in \mathscr{S}(\mathbb{R})$,

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{1}{\varepsilon}\, g\Big(\frac{t}{\varepsilon}\Big) \cdot f(x+t)\, dt \;=\; f(x)$$

This proves Fourier inversion

$$\int_{\mathbb{R}} \psi_\xi(x)\, \widehat{f}(\xi)\, d\xi \;=\; f(x) \qquad\qquad (\text{for } f \text{ Schwartz})$$

[16] The measure in the integral for Fourier inversion has no *atoms*, so in various senses would be called *continuous*, as opposed to *discrete*.

In particular, this proves that Fourier transform *bijects* the Schwartz space to itself.

[2.4] Plancherel Since the exponentials are not in $L^2(\mathbb{R})$, the abstract, elementary Hilbert-space Plancherel is essentially irrelevant here.

For f, g Schwartz, using Fourier inversion on Schwartz functions,

$$\langle \widehat{f}, \widehat{g} \rangle \;=\; \int_{\mathbb{R}} \widehat{f}(\xi) \int_{\mathbb{R}} \psi_\xi(x)\, \overline{g}(x)\, dx\, d\xi \;=\; \int_{\mathbb{R}} \overline{g}(x) \int_{\mathbb{R}} e^{2\pi i x\xi}\, \widehat{f}(\xi)\, d\xi\, dx \;=\; \int_{\mathbb{R}} f(x)\, \overline{g}(x)\, dx \;=\; \langle f, g \rangle$$

In particular, $|\widehat{f}|^2 = |f|^2$.

Now the Fourier transform is extended to $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *by continuity*, using the denseness of Schwartz functions in $L^2(\mathbb{R})$.

Expression of $f \in L^2(\mathbb{R})$ by Fourier inversion is a *spectral expansion* or *eigenfunction expansion* in the sense that it represents f as a superposition of generalized eigenfunctions for Δ. Non-convergence of the integrals on $L^2(\mathbb{R})$ is only a minor hazard, since the Fourier transform on $L^2(\mathbb{R})$ is *defined by extension* (by continuity) from the Schwartz space, where the integrals converge nicely.

[2.4.1] Remark: Even though the exponentials are not in $L^2(\mathbb{R})$, in the Plancherel formula they behave in spirit as though they were an orthonormal basis.

[2.5] Solving differential equations via spectral expansions Given $f \in L^2(\mathbb{R})$, solve the equation $(\Delta - z^2)u = f$ using the fact that Fourier transform and inversion *diagonalize* differentiation.

That is, replacing u, f by their spectral expansions,

$$(\Delta_x - z^2) \int_{\mathbb{R}} \widehat{u}(\xi)\, \psi_\xi(x)\, d\xi \;=\; \int_{\mathbb{R}} \widehat{f}(\xi)\, \psi_\xi(x)\, d\xi$$

We *must* move the differentiation inside the integral, and this is not trivially justifiable. Nevertheless, since it is inescapable, we postpone worry, and proceed:

$$\int_{\mathbb{R}} \widehat{f}(\xi)\, \psi_\xi(x)\, d\xi \;=\; \int_{\mathbb{R}} \widehat{u}(\xi)\, (\Delta_x - z^2)\psi_\xi(x)\, d\xi \;=\; \int_{\mathbb{R}} \widehat{u}(\xi)\, (-4\pi^2\xi^2 - z^2)\psi_\xi(x)\, d\xi$$

Thus, it *suffices* to take

$$\widehat{u}(\xi) \;=\; \frac{\widehat{f}(\xi)}{-4\pi^2\xi^2 - z^2}$$

For $z \notin i\mathbb{R}$, for $f \in L^2(\mathbb{R})$, the resulting $\widehat{u}$ is in $L^2(\mathbb{R})$, although we do not know *a priori* that there is $u \in L^2(\mathbb{R})$ producing $\widehat{u}$ under Fourier transform. Indeed, by Fourier inversion, there *is* corresponding $u \in L^2(\mathbb{R})$.

However, the integrals and their extensions describing Fourier transform and inversion *only converge in an L^2 sense*. There is no guarantee that the function u will be twice-differentiable in the classical pointwise sense. For this and other reasons, the sense of *differentiation* in the apparent application of Δ under the integral *cannot* be the classical pointwise sense. Rather, it is an extended sense, reasonably called L^2 *differentiation*, and best described in terms of Sobolev spaces, just below.

[2.6] Riemann-Lebesgue lemma The Riemann-Lebesgue lemma assertion that Fourier coefficients of functions on $L^2(S^1)$ go to 0 is an easy corollary of Plancherel in that case. In contrast, it is not the case that functions in $L^2(\mathbb{R})$ go to 0 pointwise, so it is not true that Fourier transforms of functions in $L^2(\mathbb{R})$ go to 0 at infinity. Of course, Fourier transforms of Schwartz functions are Schwartz, so go to 0.

A non-trivial, useful Riemann-Lebesgue lemma here is that for $u \in L^1(\mathbb{R})$, the Fourier transform $\widehat{u}$ does go to 0 at infinity. Note that the integral defining Fourier transform *does* converge for $u \in L^1(\mathbb{R})$, although some meanings will be lost since $L^1(\mathbb{R})$ is not contained in $L^2(\mathbb{R})$.

By the definition of Lebesgue integral, and by Urysohn's lemma, finite linear combinations of characteristic functions of finite intervals are *dense* in $L^1(\mathbb{R})$. With u the characteristic function of a finite interval $[a, b]$,

$$\widehat{u}(\xi) \;=\; \int_a^b \overline{\psi}_\xi(x)\,dx \;=\; \frac{e^{-2\pi i\xi b} - e^{-2\pi i\xi a}}{-2\pi i\xi}$$

Thus, for a finite linear combination u of characteristic functions of intervals, $|\widehat{u}| \ll 1/(1 + |\xi|)$. Given $v \in L^1(\mathbb{R})$, let u be a finite linear combination of characteristic functions of intervals such that $|u - v|_{L^1} < \varepsilon$. Then

$$|\widehat{u}(\xi) - \widehat{v}(\xi)| \;=\; \Big|\int_{\mathbb{R}} \overline{\psi}_\xi(x)\,u(x)\,dx \;-\; \int_{\mathbb{R}} \overline{\psi}_\xi(x)\,v(x)\,dx\Big| \;\le\; \int_{\mathbb{R}} |u(x) - v(x)|\,dx \;<\; \varepsilon$$

Since $|\widehat{u}| \ll 1/(1 + |\xi|)$, for large $|\xi|$ the absolute value $|\widehat{v}(\xi)|$ is at most 2ε. This is true for every $\varepsilon > 0$, so $\widehat{v}$ goes to 0 at infinity. ///

[2.7] Introduction to Sobolev spaces on $\mathbb{R}$

Since Fourier inversion gives eigenfunction expansions for Δ, we would *imagine* that for $u = \int_{\mathbb{R}} c(\xi)\,\psi_\xi$ in $L^2(\mathbb{R})$,

$$\Delta u \;=\; \Delta\Big(\int_{\mathbb{R}} c(\xi)\,\psi_\xi\,d\xi\Big) \;=\; \int_{\mathbb{R}} (-4\pi^2\xi^2)\cdot c_\xi\,\psi_\xi\,d\xi$$

However, convergence has been weakened, and the image cannot be in $L^2(\mathbb{R})$ for general $u \in L^2(\mathbb{R})$. Nevertheless, the resulting expansion has meaning, via Plancherel, as follows.

In terms of decrease of Fourier coefficients, the s^{th} Sobolev space is

$$H^s(\mathbb{R}) \;=\; \{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + |\xi|^2)^s \cdot |\widehat{u}(\xi)|^2\,d\xi < \infty\}$$

It is almost immediate that $H^s(\mathbb{R})$ is the completion of $\mathscr{S}(\mathbb{R})$ or of $C_c^\infty(\mathbb{R})$ with respect to the s^{th} Sobolev norm $|*|_s$ given by

$$|u|_s^2 \;=\; \int_{\mathbb{R}} (1 + |\xi|^2)^s \cdot |\widehat{u}(\xi)|^2\,d\xi$$

Then, *by design* $\Delta : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ is *continuous* when the source is given the $H^s(\mathbb{R})$ topology and the target is given the $H^{s-2}(\mathbb{R})$ topology. Extend *by continuity* to a continuous map $\Delta : H^s(\mathbb{R}) \to H^{s-2}(\mathbb{R})$.

In this context, expressions $u(x) = \int_{\mathbb{R}} c(\xi)\,\psi_\xi(x)\,d\xi$ with coefficients $c(\xi)$ of polynomial growth, such as those which might arise by applying Δ to not-strongly-convergent Fourier inversion integrals, have a natural meaning as continuous linear functionals (maps to $\mathbb{C}$) on positively-indexed Sobolev spaces, as follows.

Specifically, negatively-indexed Sobolev spaces $H^{-s}(\mathbb{R})$ are spaces of *tempered distributions*[17]

$$H^{-s}(\mathbb{R}) \;=\; \{\text{tempered distributions}\, v \;:\; \int_{\mathbb{R}} (1 + \xi^2)^{-s}\,|v(\psi_\xi)|^2 \;<\; \infty\}$$

[17] As usual, the *distributions* on $\mathbb{R}$ are the completion of $C_c^\infty(\mathbb{R})$ with respect to the weak *-topology given by semi-norms $\nu_u(v) = \int_{\mathbb{R}} u(x)\,v(x)\,dx$ for $u \in C_c^\infty(\mathbb{R})$. Another characterization is that distributions are continuous linear functionals on $C_c^\infty(R)$. The equivalence follows from the readily-provable density of $C_c^\infty(\mathbb{R})$ in its dual. *Tempered* distributions are those that extend to continuous functionals on *Schwartz* functions. These are exactly the distributions admitting a sensible Fourier transform.

Write $\widehat{v}(\xi)$ for $\widehat{v}(\psi_\xi)$. For example, *compactly supported* distributions are in $H^{-\infty}(\mathbb{R}) = \bigcup_s H^s(\mathbb{R})$.

For $u \in H^s(\mathbb{R})$ and $v \in H^{-s}(\mathbb{R})$ with $s \geq 0$, there is the natural $\mathbb{C}$-bilinear pairing

$$\langle u, v \rangle_{H^s \times H^{-s}} \;=\; \int_{\mathbb{R}} \widehat{u}(\xi) \cdot \widehat{v}(\xi)\, d\xi$$

A weighted version of Cauchy-Schwarz-Bunyakowsky proves convergence (and continuity of the pairing):

$$|\langle u, v \rangle_{H^s \times H^{-s}}| \;=\; \Big| \int_{\mathbb{R}} \widehat{u}(\xi) \cdot \widehat{v}(\xi)\, d\xi \Big| \;=\; \Big| \int_{\mathbb{R}} \Big(\widehat{u}(\xi) \cdot (1+|\xi|^2)^{s/2} \Big) \cdot \Big((1+|\xi|^2)^{-s/2} \cdot \widehat{v}(\xi) \Big)\, d\xi \Big|$$

$$\leq\; \int_{\mathbb{R}} (1+|\xi|^2)^s \cdot |\widehat{u}(\xi)|^2\, d\xi \;\;\cdot\;\; \int_{\mathbb{R}} (1+|\xi|^2)^{-s} \cdot |\widehat{v}(\xi)|^2\, d\xi \;\;=\;\; |u|_s^2 \cdot |v|_{-s}^2$$

That is, Fourier integrals not in L^2 can give continuous linear functionals on subspaces of $L^2(\mathbb{R})$ consisting of Fourier integrals converging more strongly.

The above pairing is obviously an extension of the Plancherel pairing of $L^2(\mathbb{R}) = H^0(\mathbb{R})$ with itself.

[2.7.1] Remark: Thus, differentiation inside a Fourier integral is always justified, *if* interpreted as L^2-differentiation, and the resulting integral converging in an appropriate Sobolev space. Connection to classical differentiability and continuity are best made slightly indirectly, as follows.

[2.8] Sobolev inequality/imbedding

It is natural to wonder whether functions u with sufficiently many extended-notion derivatives in $L^2(\mathbb{R})$ are differentiable in a *classical* sense. As with Fourier series, the differentiation $\Delta : H^s(\mathbb{R}) \to H^{s-2}(\mathbb{R})$ is an *extension* from the dense subspace $C_c^\infty(\mathbb{R})$, so the relation to classical differentiability is *not* tautological. [18]

In contrast to the natural Banach spaces $C^k(S^1)$, it is less clear which topological vector spaces of k-times classically differentiable functions on $\mathbb{R}$ should be compared to Sobolev spaces. There is more than a single viable option. A natural choice might appear to be spaces $C^k_{\text{bdd}}(\mathbb{R})$ of k-times continuously differentiable functions *bounded* on $\mathbb{R}$, and whose derivatives up through the k^{th} are bounded. These are Banach spaces, with norms

$$|f|_{C^k} \;=\; \sup_{0 \leq i \leq k} \sup_{x \in \mathbb{R}} |f^{(i)}(x)|$$

It is straightforward to prove that $C^o_{\text{bdd}}(\mathbb{R})$ is *complete* in the C^o-norm, so truly is a Banach space. The argument for $k > 0$ is only slightly more complicated. However, as expanded-upon below, these spaces have some pathologies. Thus, instead, take

$$C_o^k(\mathbb{R}) \;=\; \text{closure of } C_o^\infty(\mathbb{R}) \text{ with respect to } |*|_{C^k}$$

$$=\; \{k\text{-times continuously differentiable functions whose derivatives go to 0 at infinity}\}$$

The *Sobolev inequality* is that

$$|f|_{C_o^k} \;\;\ll_{s,k}\;\; |f|_{H^s} \qquad\qquad (\text{for all } s > k + \tfrac{1}{2}, \text{ for } f \in C_o^k(\mathbb{R}))$$

That is, $H^s(\mathbb{R}) \subset C_o^k(\mathbb{R})$ is a continuous map.

The simple case $k = 0$ illustrates the mechanism. The weighted version of Cauchy-Schwarz-Bunyakowsky gives

[18] In fact, here the L^2 extension of differentiation includes a finer topology on the target spaces $H^{s-2}(\mathbb{R})$ than the restriction of the usual weak $*$-topology on *tempered distributions*.

$$\Big|\sup_{x\in\mathbb{R}}\Big(\int_{\mathbb{R}} c(\xi)\,\psi_\xi(x)\,d\xi\Big)\Big| \;\le\; \sup_x \int_{\mathbb{R}} |c(\xi)|\,d\xi \;=\; \int_{\mathbb{R}} |c(\xi)|\,d\xi \;=\; \int_{\mathbb{R}} \Big(|c(\xi)|\cdot(1+|\xi|^2)^{s/2}\Big)\cdot\frac{1}{(1+|\xi|^2)^{s/2}}\,d\xi$$

$$\le\; \int_{\mathbb{R}} |c(\xi)|^2\cdot(1+|\xi|^2)^s\,d\xi \;\cdot\; \int_{\mathbb{R}}\frac{d\xi}{(1+|\xi|^2)^s} \;=\; \Big|\int_{\mathbb{R}} c(\xi)\,\psi_\xi\,d\xi\Big|^2_{H^s}\cdot\int_{\mathbb{R}}\frac{d\xi}{(1+|\xi|^2)^s}$$

For $s>\frac{1}{2}$ the latter integral is convergent. Thus, suitable Sobolev norms dominate the $C^k_o(\mathbb{R})$ norms, proving Sobolev's inequality, also called Sobolev's *imbedding*, for functions on $\mathbb{R}$.

[2.8.1] **Remark:** The same argument applies to the spaces $C^k_{\mathrm{bdd}}(\mathbb{R})$. However, the latter have pathologies fatally obstructing some reasonable goals. The issue is already clear for $k=0$, with the point that not all bounded continuous functions are *uniformly* continuous, for example, $u(x)=\sin(x^2)$. Thus, the natural map $\mathbb{R}\times C^o_{\mathrm{bdd}}(\mathbb{R})\to C^o_{\mathrm{bdd}}(R)$ by translation

$$y\times f \;\longrightarrow\; x\to f(x+y) \qquad\qquad (\text{for } x,y\in\mathbb{R} \text{ and } f\in C^o_{\mathrm{bdd}}(\mathbb{R}))$$

is *not continuous*. That is, $C^k_{\mathrm{bdd}}(\mathbb{R})$ is not a *representation space* for the natural translation action of $\mathbb{R}$. The obstacle, that bounded continuous functions on $\mathbb{R}$ need not be *uniformly* continuous, is not itself a pathology. However, natural spaces of functions should be representation spaces. Thus, the conclusion is that the spaces $C^k_{\mathrm{bdd}}(\mathbb{R})$ are not what we want.

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3. *Appendix: compact self-adjoint operators on Hilbert spaces*

[3.0.1] **Proposition:** A continuous *self-adjoint* operator T on a Hilbert space V has operator norm $|T|=\sup_{|v|\le 1}|Tv|$ expressible as

$$|T| \;=\; \sup_{|v|\le 1}|\langle Tv,v\rangle|$$

Proof: On one hand, certainly $|\langle Tv,v\rangle|\le|Tv|\cdot|v|$, giving the easy direction of inequality.

On the other hand, let $\sigma=\sup_{|v|\le 1}|\langle Tv,v\rangle|$. A polarization identity gives

$$2\langle Tv,w\rangle+2\langle Tw,v\rangle \;=\; \langle T(v+w),v+w\rangle-\langle T(v-w),v-w\rangle$$

With $w=t\cdot Tv$ with $t>0$, since $T=T^*$, both $\langle Tv,w\rangle$ and $\langle Tw,v\rangle$ are non-negative real. Taking absolute values,

$$4\langle Tv,t\cdot Tv\rangle \;=\; \Big|\langle T(v+t\cdot Tv),v+t\cdot Tv\rangle-\langle T(v-t\cdot Tv),v-t\cdot Tv\rangle\Big|$$

$$\le\; \sigma\cdot|v+t\cdot Tv|^2+\sigma\cdot|v-t\cdot Tv|^2 \;=\; 2\sigma\cdot\big(|v|^2+t^2\cdot|Tv|^2\big)$$

Divide through by $4t$ and set $t=|v|/|Tv|$ to minimize the right-hand side, obtaining

$$|Tv|^2 \;\le\; \sigma\cdot|v|\cdot|Tv|$$

giving the other inequality. ///

Recall that a continuous linear operator $T:V\to V$ is *compact* when the image of the unit ball by T has compact closure.

[3.0.2] **Theorem:** A compact self-adjoint operator T has largest eigenvalue $\pm|T|$, and an orthonormal basis of eigenfunctions. The number of eigenvalues λ larger than a given constant $c>0$ is finite. In particular, multiplicities are finite.

Proof: Suppose $|T| > 0$. Using the re-characterization of operator norm, let v_i be a sequence of unit vectors such that $|\langle Tv_i, v_i\rangle| \to |T|$. That is, the limit points of $\langle Tv_i, v_i\rangle$ are $\pm|T|$. Replace v_i by a subsequence so that $\langle Tv_i, v_i\rangle \to \lambda$ with λ one of $\pm|T|$. On one hand, using $\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}$,

$$0 \;\le\; |Tv_i - \lambda v_i|^2 \;=\; |Tv_i|^2 - 2\lambda\langle Tv_i, v_i\rangle + \lambda^2|v_i|^2 \;\le\; \lambda^2 - 2\lambda\langle Tv_i, v_i\rangle + \lambda^2$$

By assumption, the right-hand side goes to 0. Using compactness, replace v_i with a subsequence such that Tv_i has limit w. Then the inequality shows that $\lambda v_i \to w$, so $v_i \to \lambda^{-1}w$. Thus, by continuity of T, $Tw = \lambda w$.

The other assertions follow by the short natural arguments. ///

Notes

Much of the above is completely standard, and widely known for more than 100 years. Thus, much of it appears everywhere, in standard texts as well as on-line. E.g., for basic material, including spectral theory of compact operators, see course notes at http://www.math.umn.edu/~garrett/m/fun/

Eigenvalue problems in infinite-dimensional spaces, especially in Hilbert spaces, were considered prior to 1900 by Fredholm, Volterra, Hilbert, and others. Their context was usually *integral equations*, to which many differential equations were converted, to benefit from the greater tractability of the integral equations. The significance of the notion of *compact* or *completely continuous* operator for spectral theory was recognized.

Ideas about Fourier series are 200 years old, and due to several people. For technical reasons visible in the above discussion, common use of Fourier transforms is more recent, perhaps systematized only by Wiener and Bochner in the early 20th century.

Although differential operators are natural unbounded operators on Hilbert spaces, apparently the strongest motivation to discuss them in that context arose mostly from quantum physics, since the literal solution of differential equations admitted other options. Stone and von Neumann proved existence of self-adjoint extensions of arbitrary symmetric operators, but these extensions used the Axiom of Choice, and could not be *characterized* usefully. In contrast, Friedrichs' effective and explicit *construction* of a self-adjoint extension of *semi-bounded* symmetric operators allows computations.

Already in the late 19th century ideas about *energy estimates* led to use of modified inner products resembling Sobolev's constructions from the 1930s.

Although Bochner and others had thoroughly studied Fourier integrals in the early 20th century, Schwartz' systematic development of notions of *distribution* and *tempered distribution* in the late 1940s gave a useful stability to the ideas, making clear that there was more to it than a bag of tricks for regularizing integrals.

Gelfand's and Pettis' notion of *weak integral* exactly dispatches many otherwise-awkward basic analytical problems, such as differentiation in an integral with respect to a parameter. Nevertheless, that notion of integral is under-appreciated: the tradition of Riemann integrals seems to cause *strong* integrals (such as Bochner's) to be favored, for no compelling technical reason.

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