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01. Review of metric spaces

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We review basics concerning metric spaces from a contemporary viewpoint, and prove the important *Baire category theorem*, for both *complete metric* spaces and *locally compact Hausdorff*[1] spaces.

- Examples
- Metric spaces, completeness
- Completions
- Baire category theorem

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1. Examples

[1.1] Euclidean space

The prototype of a *metric space* is $\mathbb{R}^n$ with its standard distance function or *metric* $d(,)$ learned from the Pythagorean theorem:

$$d(x,y) \;=\; \sqrt{(x_1-y_1)^2+\ldots+(x_n-y_n)^2}$$

Variants also make sense:

$$d_1(x,y) \;=\; |x_1-y_1|+\ldots+|x_n-y_n| \qquad\qquad d_\infty(x,y) \;=\; \max_i |x_i-y_i|$$

The usual *ball of radius r centered at $x\in\mathbb{R}^n$* is

$$\text{ball of radius } r \text{ at } x \;=\; \{y\in\mathbb{R}^n \;:\; d(x,y)\le r\}$$

The corresponding balls for variant metrics have varying shapes: the metric $d_1(,)$ gives *squares* in $\mathbb{R}^2$, *octahedra* in $\mathbb{R}^3$, and so on. The metric $d_\infty(,)$ gives squares, cubes, tessaracts, etc.

[1.2] Discrete spaces

For an arbitrary set X, let

$$d(x,y) \;=\; \begin{cases} 1 & (\text{for } x\neq y) \\ 0 & (\text{for } x=y) \end{cases}$$

Not so interesting, but legitimate.

[1.3] Sequence spaces ℓ^p

A sort of infinite-dimensional analogue of the standard metric on $\mathbb{R}^n$ is the set

$$\ell^2 \;=\; \{x=(x_1,x_2,\ldots) \;:\; \sum_{i=1}^\infty |x_i|^2<\infty\}$$

with metric

$$d_2(x,y) \;=\; \sqrt{\sum_{i=1}^\infty |x_i-y_i|^2}$$

[1] Recall that a topological vector space is *locally compact* if every point has an open neighborhood with compact closure. A space is *Hausdorff* if for any two points x,y there are opens U,V such that $x\in U$, $y\in V$, and $U\cap V=\phi$.

In fact, replacing 2 by any $1 \le p < \infty$ works as well:

$$\ell^p \;=\; \{x = (x_1, x_2, \ldots) \;:\; \sum_{i=1}^{\infty} |x_i|^p < \infty\}$$

with metric

$$d_p(x,y) \;=\; \Big(\sum_{i=1}^{\infty} |x_i - y_i|^p\Big)^{1/p}$$

Slightly unlike the case of varying metrics on $\mathbb{R}^n$, the underlying sets ℓ^p are not the same. For example, ℓ^2 is strictly larger than ℓ^1.

[1.4] Function spaces $C^o[a,b]$

The space $C^o[a,b]$ of continuous real-valued functions on a finite interval $[a,b] \subset \mathbb{R}$ has a natural metric

$$d(f,g) \;=\; \sup_{x \in [a,b]} |f(x) - g(x)| \qquad \text{(sup-norm metric)}$$

In fact, this space is *complete* in the sense that any Cauchy sequence converges to another function in the space. That is, for a sequence $\{f_i\}$ such that, for every $\varepsilon > 0$, there is i_o such that $d(f_i, f_j) < \varepsilon$ for all $i, j \ge i_o$, then $\lim_i f_i$ is an element of $C^o[a,b]$. That limit is a function evaluated pointwise by

$$\Big(\lim_i f_i\Big)(x) \;=\; \lim_i f_i(x) \qquad \text{(for } x \in [a,b]\text{)}$$

Indeed, the pointwise limits $f(x) = \lim_i f_i(x)$ exist, since $|f_i(x) - f_j(x)| \le \sup_y |f_i(y) - f_j(y)|$ shows that the pointwise values are Cauchy sequences of real numbers. The issue is to see that the pointwise-limit function f is *continuous*.

Continuity is a natural, archetypical *three-epsilon* argument, once *uniform* continuity on $[a,b]$ is observed. Given $\varepsilon > 0$, take i_o such that $d(f_i, f_j) < \varepsilon$ for all $i, j \ge i_o$. Just take $i = i_o$. Since [2] f_i is *uniformly* continuous on $[a,b]$, take $\delta > 0$ such that $|x - y| < \delta$ implies $|f_i(x) - f_i(y)| < \varepsilon$. The triangle inequality for the usual absolute value on real numbers gives

$$\Big|f(x) - f(y)\Big| \;\le\; \Big|f(x) - f_i(x)\Big| + \Big|f_i(x) - f_i(y)\Big| + \Big|f_i(y) - f(y)\Big| \;<\; \varepsilon + \varepsilon + \varepsilon$$

This proves that the pointwise limit function f is continuous.

Resembling the standard metric on $\mathbb{R}^n$, another reasonable metric on $C^o[a,b]$ is

$$d_2(f,g) \;=\; \Big(\int_a^b |f(x) - g(x)|^2 \, dx\Big)^{\frac{1}{2}} \qquad (L^2\text{-norm metric})$$

This metric has good geometric properties, but $C^o[a,b]$ is *not* complete with this metric. For example, we can make piecewise linear continuous functions approaching the discontinuous function that is 0 on $[a, \frac{a+b}{2}]$ and 1 on $[\frac{a+b}{2}, b]$, by

$$f_n(x) \;=\; \begin{cases} 0 & (\text{for } a \le x \le \frac{a+b}{2} - \frac{1}{n}) \\ \frac{n}{2} \cdot (x - (\frac{a+b}{2} - \frac{1}{n})) & (\text{for } \frac{a+b}{2} - \frac{1}{n} \le x \le \frac{a+b}{2} + \frac{1}{n}) \\ 1 & (\text{for } \frac{a+b}{2} + \frac{1}{n} \le x \le b) \end{cases}$$

[2] The uniform continuity of continuous functions on finite closed intervals $[a,b]$ is a special case of the uniform continuity of continuous functions on *compact* sets in metric spaces.

(Draw a picture.) The pointwise limit is 0 to the left of the midpoint, and 1 to the right. Despite the fact that the pointwise limit does not exist at the midpoint,

$$d_2(f_i, f_j)^2 \;\le\; \int_{\frac{a+b}{2}-\frac{1}{n}}^{\frac{a+b}{2}+\frac{1}{n}} 1\,dx \;\le\; \frac{2}{n} \qquad\qquad (\text{for } i,j \ge n)$$

which goes to 0 as $n \to \infty$.

[1.5] Function spaces $C^1[a,b]$

The space $C^1[a,b]$ is differentiable functions with continuous derivatives on $[a,b]$, that is, *continuously once-differentiable*. This space has a natural metric

$$d(f,g) \;=\; \sup_{x\in[a,b]} |f(x)-g(x)| \;+\; \sup_{x\in[a,b]} |f'(x)-g'(x)|$$

This space *is* complete: Cauchy sequences of functions have limits which are again continuously once-differentiable. The proof has some subtlety, and we postpone it. A different point is that

$$\frac{d}{dx} \;:\; C^1[a,b] \;\longrightarrow\; C^o[a,b]$$

is a *continuous* linear map.

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2. *Metric spaces, continuous maps, completeness*

[2.1] Abstract notion of metric space

The abstracted idea of a *metric space* is of a set with a notion of *distance* on it,

$$d(x,y) \;=\; \text{distance from } x \text{ to } y$$

which should conform to basic physical intuition about distance. Thus, first, the only point y at distance 0 from a point x is $y = x$ itself. Second, distance does not depend on *direction*: the distance from x to y is the same as the distance from y to x. Third, thinking that the distance from x to y should correspond to a *shortest route* of some kind from x to y, the distance from x to y should be at most the sum of distances from x to any other intermediate point z and then from z to y, giving the *triangle inequality*.

That is, a *metric space* X, d is a set X with a *metric* $d(,) : X \times X \to \mathbb{R}$, a real-valued function such that, for $x, y, z \in X$,
- *(Positivity)* $d(x,y) \ge 0$ and $d(x,y) = 0$ if and only if $x = y$
- *(Symmetry)* $d(x,y) = d(y,x)$
- *(Triangle inequality)* $d(x,z) \le d(x,y) + d(y,z)$

A *Cauchy sequence* in a metric space X is a sequence $x_1, x_2, \ldots$ with the property that for every $\varepsilon > 0$ there is N sufficiently large such that for $i,j \ge N$ we have $d(x_i, x_j) < \varepsilon$. A point $x \in X$ is a **limit** of that Cauchy sequence if for every $\varepsilon > 0$ there is N sufficiently large such that for $i \ge N$ we have $d(x_i, x) < \varepsilon$. A subset X of a metric space Y is **dense** in Y if every point in Y is a limit of a Cauchy sequence in X.

[2.2] Continuous maps

Generalizing the usual discussion of continuity on $\mathbb{R}^n$, a map $f : X \to Y$ from one metric space to another is *continuous* at $x \in X$ if, for every $\varepsilon > 0$, there is $\delta > 0$ such that $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$.

The map $f : X \to Y$ is *continuous* if it is continuous at every point of X.

[2.2.1] Proposition: A map $f : X \to Y$ continuous if and only if, for every Cauchy sequence $\{x_i\}$ in X converging to $x \in X$,

$$\lim_i f(x_i) \;=\; f\big(\lim_i x_i\big)$$

Proof: On one hand, for continuous f and $x_i \to x$, given any ball at x, from some point onward all the x_i are inside that ball. Thus, eventually, $f(x_i)$ is close to $f(x)$, so $\lim_i f(x_i) = f(x)$.

On the other hand, if f were *not* continuous at x, then for some $\varepsilon > 0$, for every $\delta > 0$ there is some x' inside the δ-ball at x such that $|f(x) - f(x')| \geq \varepsilon$. Take the sequence of δ's given by $\delta_n = 1/n$, and let x_n be the corresponding bad x'. Then $x_n \to x$ but $f(x_n) \not\to f(x)$, so f is not continuous at x. ///

[2.2.2] Remark: The property of the proposition is *sequential* continuity. It is necessary to assume explicitly that $\{x_i\}$ has a limit $x \in X$, or else make the blanket assumption that X is *complete*, as just below.

[2.3] Completeness

A metric space is *complete* if every Cauchy sequence has a limit. [3] Unsurprisingly, *complete* metric spaces have the advantage that *infinitary* operations often make sense.

The following standard lemma is often useful, and makes explicit a bit of intuition.

[2.3.1] Lemma: Let $\{x_i\}$ be a Cauchy sequence in a metric space X, d, and suppose that the sequence converges to x in X. Given $\varepsilon > 0$, let N be sufficiently large such $d(x_i, x_j) < \varepsilon$ for $i, j \geq N$. Then $d(x_i, x) \leq \varepsilon$ for $i \geq N$.

Proof: Let $\delta > 0$ and take $j \geq N$ also large enough such that $d(x_j, x) < \delta$. Then for $i \geq N$ by the triangle inequality

$$d(x_i, x) \leq d(x_i, x_j) + d(x_j, x) < \varepsilon + \delta$$

Since this holds for every $\delta > 0$ we have the result. ///

[2.4] Metric topologies

A metric space X has a *natural topology* with *basis* given by *open balls*

$$\{y \in X : d(x, y) < r\} \;=\; (\text{open ball of radius } r > 0 \text{ centered at } x \in X)$$

That is, a set $U \subset X$ is *open* when around every point $x \in U$ there is an open ball of positive radius contained in U.

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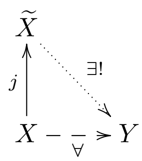
[3] Convergence of Cauchy *sequences* is more properly called *sequential completeness.* In fact, for metric spaces, sequential completeness *implies* implies the strongest form of completeness, namely convergence of Cauchy *nets*, as we will observe more carefully later. This is *not* so important at the moment, but will have some importance for non-metrizable spaces, which are *rarely* complete (in the strongest sense), but in practice often are at least sequentially complete. A useful form of completeness stronger than sequential completeness but weaker than outright completeness is *local completeness*, also called *quasi-completeness*, which will play a significant role later.

3. *Completions*

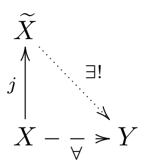
A not-complete metric space presents the difficulty that Cauchy sequences may fail to converge. Reasonably, we want to repair this situation, and in as economical way as possible. [4]

Our *intention* is that, when a metric space X, d_X is not complete, there should be a *complete* metric space $\widetilde{X}, d_{\widetilde{X}}$ and *isometry* (distance-preserving) $j : X \to \widetilde{X}$, such that every continuous $f : X \to Y$ to *complete* metric space Y *factors through* j uniquely. That is, there are *commutative diagrams*[5] of continuous maps



(for every continuous $X \to Y$???)

By chance, this characterization accidentally asks impossibly much of the completion. The basic problem is that completeness is not a *topological property*, but metric, insofar as merely-continuous maps can fail to map Cauchy sequences to Cauchy sequences. [6] That is, the universal property could only plausibly be guaranteed for some restricted class of continuous functions $f : X \to Y$ preserving Cauchy sequences. An obvious class is *non-expanding* f, meaning $d_Y(f(x), f(x')) \le d_X(x, x')$ for all $x, x' \in X$. A somewhat weaker condition on f that promises preservation of Cauchy sequences is *uniform* continuity of $f : X \to Y$, meaning that, for any $\varepsilon > 0$, there is $\delta > 0$ such that, for all $x, x' \in X$, $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$. Uniformly continuous maps preserve Cauchy sequences. [7] Thus, one adequately corrected condition is



(for every *uniformly* continuous $X \to Y$)

Such an $\widetilde{X}$ is a *completion* of X. Thus, we *characterize* completions by certain of their interaction with other metric spaces, explaining our expectations of them, rather than giving a construction with unclear motivations and properties.

We will prove that completions *exist* (by constructing one) and are essentially *unique*.

Before describing any *constructions* of completions, we can prove some things about the behavior of any *possible* completion. In particular, we will prove that any two completions are *naturally isometric* to each other. *Thus, the outcome will be independent of construction.*

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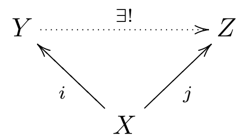
[4] One traditional approach is to immediately *construct* a *completion* as equivalence class of Cauchy sequences in X. This has the disadvantage that it does not address the possibility of other constructions yielding different outcomes!

[5] A diagram of maps is *commutative* when the composite map from one object to another within the diagram does not depend on the route taken within the diagram.

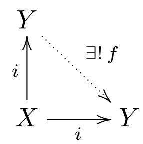
[6] The interval $[1, +\infty)$ and the half-open interval $(0, 1]$ are homeomorphic by $x \to 1/x$, but with the usual metric the former is complete while the latter is not. The Cauchy sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ in $(0, 1]$ maps back to the non-Cauchy sequence $1, 2, 3, \ldots$ in $[1, \infty)$.

[7] For *uniformly* continuous $f : X \to Y$ and Cauchy $\{x_i\}$ in X, given $\varepsilon > 0$, choose $\delta > 0$ so that $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \varepsilon$. Choose N large enough so that $d_X(x_i, x_j) < \delta$ for $i, j \ge N$. Then $d_Y(f(x_i), f(x_j)) < \varepsilon$ for all $i, j \ge N$, so $\{f(x_i)\}$ is Cauchy in Y.

[3.0.1] Proposition: *(Uniqueness)* Let $i : X \to Y$ and $j : X \to Z$ be two completions of a metric space X. Then there is a unique homeomorphism $h : Y \to Z$ such that $j = h \circ i$. That is, we have commutative diagram

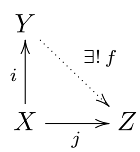


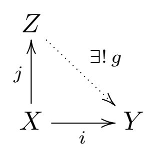
Proof: First, take $Y = Z$ and $f : X \to Y$ to be the inclusion i, in the characterization of $i : X \to Y$. The defining characterization of $i : X \to Y$ shows that there is *unique* continuous $f : Y \to Y$ fitting into a commutative diagram



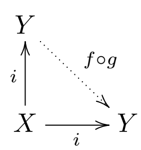
Since the *identity* map $Y \to Y$ certainly fits into this diagram, the *only* continuous map f fitting into the diagram is the identity on Y.

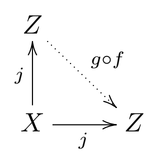
Next, applying the characterizations of both $i : X \to Y$ and $j : X \to Z$, we have unique continuous $f : Y \to Z$ and $g : Z \to Y$ fitting into





Then $f \circ g : Y \to Y$ and $g \circ f : Z \to Z$ fit into





By the first observation, this means that $f \circ g$ is the identity on Y, and $g \circ f$ is the identity on Z, so f and g are mutual inverses, and Y and Z are *homeomorphic*. ///

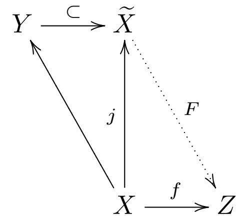
[3.0.2] Remark: A virtue of the characterization of completion is that it does not refer to the *internals* of any completion.

Next, prove that the mapping-property characterization of a completion does not introduce more points than absolutely necessary:

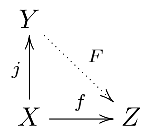
[3.0.3] Proposition: Every point in a completion $\widetilde{X}$ of X is the limit of a Cauchy sequence in X. That is, X is *dense* in $\widetilde{X}$.

Proof: Write $d(,)$ for both the metric on X and its extension to $\widetilde{X}$. Let $Y \subset \widetilde{X}$ be the collection of limits of Cauchy sequences of points in X. We claim that Y itself is *complete*. Indeed, given a Cauchy sequence $\{y_i\}$ in Y with limit $z \in \widetilde{X}$, let $x_i \in X$ such that $d(x_i, y_i) < 2^{-i}$. It will suffice to show that $\{x_i\}$ is Cauchy with limit z. Indeed, given $\varepsilon > 0$, take N large enough so that $d(y_i, z) < \varepsilon/2$ for all $i \geq N$, and increase N if necessary so that $2^{-i} < \varepsilon/2$. Then, by the triangle inequality, $d(x_i, z) < \varepsilon$ for all $i \geq N$. Thus, Y is complete.

By the defining property of $\widetilde{X}$, every uniformly continuous $f : X \to Z$ to complete Z has a unique extension to a continuous $F : \widetilde{X} \to Z$ fitting into



Since Y is already complete and $j(X) \subset Y$, the restriction of F to Y gives a diagram



That is, Y fits the characterization of a completion of X. By uniqueness, $Y \subset \widetilde{X}$ is a homeomorphism, so $Y = \widetilde{X}$. ///

Now prove *existence* of completions by giving the standard *construction*.

The intention is that every Cauchy sequence has a limit, so we should (somehow!) *adjoin* points as needed for these limits. However, different Cauchy sequences may happen to have the same limit.

Thus, we want an equivalence relation on Cauchy sequences that says they should have the same limit, even without knowing the limit exists or having somehow constructed or adjoined the limit point.

Define an equivalence relation $\sim$ on the set C of Cauchy sequences in X, by

$$\{x_s\} \sim \{y_t\} \iff \lim_s d(x_s, y_s) = 0$$

Attempt to define a metric on the set $C/\sim$ of equivalence classes by

$$d(\{x_s\}, \{y_t\}) = \lim_s d(x_s, y_s)$$

We must verify that this is well-defined on the quotient $C/\sim$ and gives a metric. We have an injection $j : X \to C/\sim$ by

$$x \to \{x, x, x, \ldots\} \bmod \sim$$

[3.0.4] Proposition: $j : X \to C/\sim$ is a completion of X.

Proof: Grant for the moment that the distance function on $\widetilde{X} = C/\sim$ is well-defined, and is complete, and show that it has the property of a completion of X. To this end, let $f : X \to Y$ be a *uniformly* continuous map to a complete metric space Y.

Given $z \in \widetilde{X}$, choose a Cauchy sequence x_k in X with $j(x_k)$ converging to z, and *try* to define $F : \widetilde{X} \to Y$ in the natural way, by

$$F(z) = \lim_k f(x_k)$$

Since f is *uniformly* continuous, $f(x_k)$ is Cauchy in Y, and by completeness of Y has a limit, so $F(z)$ *exists*, at least if well-defined.

For well-definedness of $F(z)$, for x_k and x'_k two Cauchy sequences whose images $j(x_k)$ and $j(x'_k)$ approach z, since j is an isometry eventually x_k is close to x'_k, so $f(x_k)$ is eventually close to $f(x'_k)$ in Y, showing $F(z)$ is well-defined.

We saw that every element of $\widetilde{X}$ is a limit of a Cauchy sequence $j(x_k)$ for x_k in X, and *any* continuous $\widetilde{X} \to Y$ respects limits, so F is the only possible extension of f to $\widetilde{X}$.

The obvious argument will show that F is continuous. Namely, let $z, z' \in \widetilde{X}$, with Cauchy sequences x_t and x'_t approaching z and z'. Given $\varepsilon > 0$, by uniform continuity of F, there is N large enough such that $d_Y(F(j(x_r)), F(j(x_s))) < \varepsilon$ and $d_Y(F(j(x'_r)), F(j(x'_s))) < \varepsilon$ for $r, s \geq N$. From the lemma above (!), for such r even in the limit the strict inequalities are at worst non-strict inequalities:

$$d_Y(f(x_r), F(z)) \leq \varepsilon \qquad \text{and} \qquad d_Y(f(x'_r), F(z')) \leq \varepsilon$$

By the triangle inequality, since $f : X \to Y$ is continuous, we can increase r to have $d_X(x_r, x'_r)$ small enough so that $d_Y(f(x_r), f(x'_r)) < \varepsilon$, and then

$$d_Y(F(z), F(z')) \leq d_Y(F(z), f(x_r)) + d_Y(f(x_r), f(x'_r)) + d_Y(f(x'_r), F(z')) \leq \varepsilon + \varepsilon + \varepsilon$$

Since $j : X \to \widetilde{X}$ is an isometry,

$$d_X(x_r, x'_r) = d_{\widetilde{X}}(j(x_r), j(x'_r)) \leq d_{\widetilde{X}}(j(x_r), z) + d_{\widetilde{X}}(z, z') + d_{\widetilde{X}}(j(x'_r), z')$$

so

$$d_X(x_r, x'_r) \leq d_{\widetilde{X}}(z, z') + 2\varepsilon$$

Thus,

$$d_Y(F(z), F(z')) \leq d_{\widetilde{X}}(z, z') + 4\varepsilon \qquad \text{(for all } \varepsilon > 0\text{)}$$

Thus, F is continuous. Granting that $\widetilde{X} = C/\sim$ is complete, etc., it is a completion of X.

It remains to prove that the apparent metric on $\widetilde{X}$ truly is a metric, and that $\widetilde{X}$ is complete.

First, the limit in attempted definition

$$d(\{x_s\}, \{y_t\}) = \lim_s d(x_s, y_s)$$

does exist: given $\varepsilon > 0$, take N large enough so that $d(x_i, x_j) < \varepsilon$ and $d(y_i, y_j) < \varepsilon$ for $i, j \geq N$. By the triangle inequality,

$$d(x_i, y_i) \leq d(x_i, x_N) + d(x_N, y_N) + d(y_N, y_i) < \varepsilon + d(x_N, y_N) + \varepsilon$$

Similarly,

$$d(x_i, y_i) \geq -d(x_i, x_N) + d(x_N, y_N) - d(y_N, y_i) > -\varepsilon + d(x_N, y_N) - \varepsilon$$

Thus, unsurprisingly,

$$\Big| d(x_i, y_i) - d(x_N, y_N) \Big| < 2\varepsilon$$

and the sequence of real numbers $d(x_i, y_i)$ is Cauchy, so convergent.

Similarly, when $\lim_i d(x_i, y_i) = 0$, then $\lim_i d(x_i, z_i) = \lim_i d(y_i, z_i)$ for any other Cauchy sequence z_i, so the distance function is *well-defined* on $C/\sim$.

The positivity and symmetry for the alleged metric on $C/\sim$ are immediate. For triangle inequality, given x_i, y_i, z_i and $\varepsilon > 0$, let N be large enough so that $d(x_i, x_j) < \varepsilon$, $d(y_i, y_j) < \varepsilon$, and $d(z_i, z_j) < \varepsilon$ for $i, j \geq N$. As just above,

$$\Big| d(\{x_s\}, \{y_s\}) - d(x_i, y_i) \Big| < 2\varepsilon$$

Thus,

$$d(\{x_s\}, \{y_s\}) \leq 2\varepsilon + d(x_N, y_N) \leq 2\varepsilon + d(x_N, z_N) + d(z_N, y_N) \leq 2\varepsilon + d(\{x_s\}, \{z_s\}) + 2\varepsilon + d(\{z_s\}, \{y_s\}) + 2\varepsilon$$

This holds for all $\varepsilon > 0$, so we have the triangle inequality.

Finally, perhaps anticlimactically, the completeness. Given Cauchy sequences $c_s = \{x_{sj}\}$ in X such that $\{c_s\}$ is Cauchy in $C/\sim$, for each s we will choose large-enough $j(s)$ such that the diagonal-ish sequence $y_\ell = x_{\ell,j(\ell)}$ is a Cauchy sequence in X to which $\{c_s\}$ converges.

Given $\varepsilon > 0$, take i large enough so that $d(c_s, c_t) < \varepsilon$ for all $s, t \geq i$. For each i, choose $j(i)$ large enough so that $d(x_{ij}, x_{ij'}) < \varepsilon$ for all $j, j' \geq j(i)$. Let $c = \{x_{i,j(i)} : i = 1, 2, \ldots\}$. For $s \geq i$,

$$d(c_s, c) \;=\; \lim_\ell d(x_{s\ell}, x_{\ell,j(\ell)}) \;\leq\; \sup_{\ell \geq i} d(x_{s\ell}, x_{\ell,j(\ell)}) \;\leq\; \sup_{\ell \geq i} \Big(d(x_{s\ell}, x_{s,j(\ell)}) + d(x_{s,j(\ell)}, x_{\ell,j(\ell)}) \Big) \;\leq\; 2\varepsilon$$

This holds for all $\varepsilon > 0$, so $\lim_s c_s = c$, and $C/\sim$ is complete. ///

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4. *The Baire category theorem*

This standard result is both indispensable and mysterious.

A set E in a topological space X is *nowhere dense* if its closure $\bar{E}$ contains no non-empty open set. A *countable union* of nowhere dense sets is said to be *of first category*, while every other subset (if any) is *of second category*. The idea (not at all clear from this traditional terminology) is that first category sets are *small*, while second category sets are *large*. In this terminology, the theorem's assertion is equivalent to the assertion that (non-empty) *complete metric* spaces and *locally compact Hausdorff* spaces are *of second category*.

Further, a G_δ set is a countable intersection of open sets. Concommitantly, an F_σ set is a countable union of closed sets. Again, the following theorem can be paraphrased as asserting that, in a complete metric space, *a countable intersection of dense G_δ's is still a dense G_δ.*

[4.0.1] Theorem: *(Baire category)* Let X be a set with metric d making X a *complete metric space*. Or let X be a locally compact Hausdorff topological space. The intersection of a *countable* collection $U_1, U_2, \ldots$ of *dense open subsets* U_i of X is still *dense* in X.

Proof: Let B_o be a non-empty open set in X, and show that $\bigcap_i U_i$ meets B_o. Suppose that we have inductively chosen an open ball B_{n-1}. By the denseness of U_n, there is an open ball B_n whose closure $\overline{B_n}$ satisfies

$$\overline{B_n} \subset B_{n-1} \cap U_n$$

Further, for complete metric spaces, take B_n to have radius less than $1/n$ (or any other sequence of reals going to 0), and in the locally compact Hausdorff case take B_n to have compact closure.

Let

$$K \;=\; \bigcap_{n \geq 1} \overline{B_n} \;\subset\; B_o \cap \bigcap_{n \geq 1} U_n$$

For complete metric spaces, the centers of the nested balls B_n form a Cauchy sequence (since they are nested and the radii go to 0). By completeness, this Cauchy sequence *converges*, and the limit point lies inside each *closure* $\overline{B_n}$, so lies in the intersection. In particular, K is non-empty. For locally compact Hausdorff spaces, the intersection of a nested family of non-empty compact sets is non-empty, so K is non-empty, and B_o necessarily meets the intersection of the U_n. ///
