(*January 11, 2013*)

Introduction to Levi-Sobolev spaces

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The simplest case of a Levi-Sobolev *imbedding theorem* asserts that the $+1$-index Levi-Sobolev space $H^1[a,b]$ (below) is inside $C^o[a,b]$. This is a corollary of a Levi-Sobolev *inequality* asserting that the $C^o[a,b]$ norm is *dominated* by the $H^1[a,b]$ norm. All that is used is the fundamental theorem of calculus and the Cauchy-Schwarz-Bunyakowsky inequality. The point is that there is a large *Hilbert-space* $H^1[a,b]$ (below) inside the *Banach* space $C^o[a,b]$.

Let

$$L^2[a,b] \;=\; \text{completion of } \; C^o[a,b] \text{ with respect to } |f|_{L^2} = \Big(\int_a^b |f(t)|^2\,dt\Big)^{1/2}$$

The $+1$-index *Levi-Sobolev* space [1] $H^1[a,b]$ is

$$H^1[a,b] \;=\; \text{completion of } \; C^1[a,b] \text{ with respect to } |f|_{H^1} = \Big(|f|^2_{L^2[a,b]} + |f'|^2_{L^2[a,b]}\Big)^{1/2}$$

[1.0.1] Theorem: *(Levi-Sobolev inequality)* On $C^1[a,b]$, the $H^1[a,b]$-norm *dominates* the $C^o[a,b]$-norm. That is, there is a constant C depending only on a,b such that $|f|_{C^o[a,b]} \le C\cdot|f|_{H^1[a,b]}$ for every $f\in C^1[a,b]$.

Proof: For $a\le x\le y\le b$, for $f\in C^1[a,b]$, the fundamental theorem of calculus gives

$$|f(y)-f(x)| \;=\; \Big|\int_x^y f'(t)\,dt\Big| \;\le\; \int_x^y |f'(t)|\,dt \;\le\; \Big(\int_x^y |f'(t)|^2\,dt\Big)^{1/2}\cdot\Big(\int_x^y 1\,dt\Big)^{1/2}$$

$$\le\; |f'|_{L^2}\cdot|x-y|^{\frac{1}{2}} \;\le\; |f'|_{L^2}\cdot|a-b|^{\frac{1}{2}}$$

Using the continuity of $f\in C^1[a,b]$, let $y\in[a,b]$ be such that $|f(y)| = \min_x |f(x)|$. Using the previous inequality,

$$|f(x)| \;\le\; |f(y)| + |f(x)-f(y)| \;\le\; \frac{\int_a^b |f(t)|\,dt}{|a-b|} \;+\; |f(x)-f(y)| \;\le\; \frac{\int_a^b |f|\cdot 1}{|a-b|} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac{1}{2}}$$

$$\le\; \frac{|f|_{L^2}^{\frac{1}{2}}\cdot|a-b|^{\frac{1}{2}}}{|a-b|} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac{1}{2}} \;=\; \frac{|f|_{L^2}^{\frac{1}{2}}}{|a-b|^{\frac{1}{2}}} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac{1}{2}} \;\le\; \Big(|f|_{L^2}+|f'|_{L^2}\Big)\cdot\Big(|a-b|^{-\frac{1}{2}}+|a-b|^{\frac{1}{2}}\Big)$$

$$\le\; 2\big(|f|^2+|f'|^2\big)^{1/2}\cdot\Big(|a-b|^{-\frac{1}{2}}+|a-b|^{\frac{1}{2}}\Big) \;=\; |f|_{H^1}\cdot 2\Big(|a-b|^{-\frac{1}{2}}+|a-b|^{\frac{1}{2}}\Big)$$

Thus, on $C^1[a,b]$ the H^1 norm dominates the C^o-norm. ///

[1.0.2] Corollary: *(Levi-Sobolev imbedding)* $H^1[a,b]\subset C^o[a,b]$.

Proof: Since $H^1[a,b]$ is the H^1-norm completion of $C^1[a,b]$, every $f\in H^1[a,b]$ is an H^1-limit of functions $f_n\in C^1[a,b]$. That is, $|f-f_n|_{H^1[a,b]}\to 0$. Since the H^1-norm dominates the C^o-norm, $|f-f_n|_{C^o[a,b]}\to 0$. A C^o limit of continuous functions is continuous, so f is continuous. ///

[1.0.3] Corollary: (of proof of theorem) $|f(x)-f(y)|\le|f'|_{L^2}\cdot|x-y|^{\frac{1}{2}}$ for $f\in H^1[a,b]$. ///

[1] ... also denoted $W^{1,2}[a,b]$, where the superscript 2 refers to L^2, rather than L^p. Beppo Levi noted the importance of taking Hilbert space completion with respect to this norm in 1906. Sobolev's work was in in the mid-1930's.

[1.0.4] Remark: That is, once we know that $H^1[a,b] \subset C^o[a,b]$, the proof of the theorem gives a stronger conclusion than mere continuity, although not as strong as differentiability.
