(*October 18, 2012*)

Preview of vector-valued integrals

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

Rather than *constructing* integrals as limits following [Bochner 1935], [Birkhoff 1935], *et alia*, we use the [Gelfand 1936]-[Pettis 1938] *characterization* of integrals. Existence is proven separately.

Existence is provable for vectorspaces with adequate completeness properties. [1]

The immediate application is to *differentiation inside an integral* with respect to a parameter.

---

1. *Gelfand-Pettis (weak) integrals*

Let V be a topological vectorspace. For a continuous V-valued function f on a measure space X a *Gelfand-Pettis integral* of f is $I_f \in V$ such that

$$\lambda(I_f) \;=\; \int_X \lambda \circ f \qquad\qquad (\text{for all } \lambda \in V^*)$$

When it exists and is unique, this vector I_f would be denoted by

$$I_f \;=\; \int_X f \;=\; \int_X f(x)\, dx$$

In contrast to *construction* of integrals as limits of Riemann sums, the Gelfand-Pettis *characterization* is a property no reasonable notion of integral would lack. Since this property is an irreducible minimum, this definition of integral is called a *weak integral*.

Uniqueness of the integral is immediate when the dual V^* *separates points*, meaning that for $v \neq v'$ in V there is $\lambda \in V^*$ with $\lambda v \neq \lambda v'$. This separation property certainly holds for Hilbert spaces: the map $\lambda w = \langle w, v - v' \rangle$ is a continuous linear functional and $\lambda(v - v') \neq 0$ gives $\lambda v \neq \lambda v'$. The separation property for Banach spaces is part of the *Hahn-Banach theorem*. [2] [3] For the rest of this discussion, all topological vector spaces are assumed locally convex without further mention.

---

[1] Precisely, things work out fine for *quasi-complete*, locally convex topological vectorspaces. This includes Hilbert, Banach, and Fréchet spaces, as well as *LF spaces*: strict colimits of Fréchet, such as $C^o_c(\mathbb{R})$ and $C^\infty_o(\mathbb{R})$). Also included are these spaces' weak-star *duals*, and other spaces of mappings such as the *strong operator topology* on mappings between Hilbert spaces, in addition to the *uniform* operator topology.

[2] In fact, Hahn-Banach holds for all *locally convex* topological vector spaces, that is, topological vector space with a local basis at 0 consisting of *convex* sets. This includes Fréchet spaces, strict colimits of Fréchet spaces such as $C^o_c(\mathbb{R})$ or $C^\infty_c(\mathbb{R})$, dual spaces of these, and essentially every reasonable space.

[3] Although every reasonable topological vector space is locally convex, it is not difficult to construct topological vector spaces which *fail* to possess this property. The spaces ℓ^p with $0 < p < 1$ are simple examples, whose main utility is illustrating the possibility of failure of local convexity.

Similarly, *linearity* of $f \to I_f$ follows when V^* separates points. Thus, the issue is proving *existence.* [4]

We integrate nice functions: compactly-supported and continuous, on measure spaces with *finite, positive, Borel* measures. In this situation, all the $\mathbb{C}$-valued integrals

$$\int_X \lambda \circ f \;=\; \int_X \lambda\big(f(x)\big)\,dx$$

exist for elementary reasons, being integrals of compactly-supported $\mathbb{C}$-valued continuous functions on a compact set with respect to a finite Borel measure.

The crucial requirement on V is that *the convex hull of a compact set has compact closure.*

It is not too hard to show that Hilbert, Banach, or Fréchet spaces have this property, because of their *completeness.*

However, non-metrizable spaces need a subtler notion of completeness, namely, *quasi-completeness*, meaning that *bounded* Cauchy *nets* converge. [5] In all applications, when the compactness of closures of convex hulls of compacts holds, it seems that the space is quasi-complete. Thus, while *a priori* the condition of quasi-completeness is stronger than the compactness condition, no example of a strict comparison seems immediate.

[1.0.1] Theorem: Let X be a compact Hausdorff topological space with a *finite*, positive, Borel measure. Let V be a locally convex topological vectorspace in which the *closure of the convex hull of a compact set is compact.* Then *continuous compactly-supported* V-valued functions f on X have Gelfand-Pettis integrals. Further,

$$\int_X f \;\in\; \text{meas}\,(X)\cdot\Big(\text{closure of convex hull of } f(X)\Big) \qquad\qquad (\textit{Proof later.})$$

[1.0.2] Remark: The conclusion that the integral of f lies in the closure of a convex hull, is a substitute for the estimate of a $\mathbb{C}$-valued integral by the integral of its absolute value.

---

2. *The mapping property*

Let X be a compact, Hausdorff, compact topological space with a positive, regular Borel measure.

Let $T : V \to W$ be a continuous linear map of locally convex, quasi-complete topological vector spaces.

[2.0.1] Corollary: For a continuous V-valued function f on X,

$$T\Big(\int_X f\Big) \;=\; \int_X T\circ f$$

Proof: The right-hand side is the Gelfand-Pettis integral of the continuous, compactly-supported W-valued function $T \circ f$, while the left-hand side is the image under T of the Gelfand-Pettis integral of f.

---

[4] We do require that the integral of a V-valued function be in the space V itself, rather than in a larger space containing V, such as a double dual V^{**}, for example. Some discussions of integration do allow integrals to exist in larger spaces.

[5] In topological vectorspaces lacking countable local bases, quasi-completeness is more relevant than completeness. For example, the weak $*$-dual of an infinite-dimensional Hilbert space is *never* complete, but is always quasi-complete. This example is non-trivial.

Since W^* separates points, the equality will follow from proving that

$$\mu\Big(T(\int_X f)\Big) \;=\; \mu\Big(\int_X T\circ f\Big) \qquad \text{(for all } \mu\in W^*)$$

Noting that $\mu\circ T\in V^*$, from the characterization of the Gelfand-Pettis integrals,

$$\mu\Big(T(\int_X f)\Big) \;=\; (\mu\circ T)\Big(\int_X f\Big) \;=\; \int_X (\mu\circ T)f \;=\; \int_X \mu(T\circ f) \;=\; \mu\Big(\int_X T\circ f\Big)$$

as desired. ///

---

3. *Differentiation under the integral*

Recall that the maps $\frac{d}{dx}: C^k(S^1)\to C^(S^1)$ and $\frac{d}{dx}: C^\infty(S^1)\to C^\infty(S^1)$ are *continuous*.

[3.0.1] Lemma: For $F\in V$ with either $V=C^k(S^1)$ or $V=C^\infty(S^1)$, the V-valued function

$$f(y) \;=\; \Big(x\longrightarrow F(x+y)\Big)$$

is a *continuous* V-valued function of $y\in S^1$.

Proof: For $V=C^k(S^1)$, we must show that, given $y_o\in S^1$ and given $\varepsilon>0$, there is $\delta>0$ such that for $|y_o-y|<\delta$

$$\|f(y_o)-f(y)\|_V \;<\; \varepsilon \qquad \text{(for } |y_o-y|<\delta)$$

That is, we must show that

$$\sup_{x\in S^1} |F^{(\ell)}(x+y_o)-F^{(\ell)}(x+y)| \;<\; \varepsilon \qquad \text{(for all } 0\le\ell\le k \text{ and } |y_o-y|<\delta)$$

Since S^1 is compact, the finitely-many continuous functions $F^{(\ell)}$ are *uniformly* continuous, so there exists such a δ.

The analogous continuity assertion for $V=C^\infty(S^1)$ follows from the aggregate of assertions about $C^k(S^1)$, and from the earlier observation that a basis of neighborhoods at 0 in $C^\infty(S^1)$ is given by intersection with bases of neighborhoods in all the $C^k(S^1)$'s. That is, continuity of the $C^\infty(S^1)$-valued function follows from the continuity in all $C^k(S^1)$ topologies. ///

[3.0.2] Corollary: For $\varphi\in C^o(S^1)$ and for $F\in C^1(S^1)$,

$$\frac{d}{dx}\int_{S^1} F(x+y)\,\varphi(y)\,dy \;=\; \int_{S^1} F'(x+y)\,\varphi(y)\,dy$$

Proof: The function-valued function $y\to F(x+y)$ was just shown continuous as $C^1(S^1)$-valued function. Multiplying by the continuous scalar-valued function $\varphi(y)$ does not disturb continuity. Thus, $y\to F(x+y)\cdot\varphi(y)$ is a continuous $C^1(S^1)$-valued function.

Thus, this function has a Gelfand-Pettis integral. From above, the continuous map $T=\frac{d}{dx}$ from $C^1(S^1)$ to $C^o(S^1)$ commutes with the Gelfand-Pettis integrals, giving the equality. ///

Bibliography

[Birkhoff 1935] G. Birkhoff, *Integration of functions with values in a Banach space*, Trans. AMS **38** (1935), 357-378.

[Bochner 1935] S. Bochner, *Integration von Funktionen deren Werte die Elemente eines Vektorraumes sind*, Fund. Math., vol. 20, 1935, pp. 262-276.

[Gelfand 1936] I. M. Gelfand, *Sur un lemme de la theorie des espaces lineaires*, Comm. Inst. Sci. Math. de Kharkoff, no. 4, vol. 13, 1936, pp. 35-40.

[Pettis 1938] B. J. Pettis, *On integration in vector spaces*, Trans. AMS, vol. 44, 1938, pp. 277-304.

[Phillips 1940] R. S. Phillips, *Integration in a convex linear topological space*, Trans. AMS, vol. 47, 1940, pp. 114-145.

[Taylor 1938] A. E. Taylor, *The resolvent of a closed transformation*, Bull. AMS, vol. 44, 1938, pp. 70-74.