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Banach-Alaoglu, boundedness, weak-to-strong principles

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- Banach-Alaoglu: compactness of polars
- Variant Banach-Steinhaus/uniform boundedness
- Second polars
- Weak boundedness implies boundedness
- Weak-to-strong differentiability

The comparison of *weak* and *strong* differentiability is due to Grothendieck, although the original sources are not widely available.

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1. *Banach-Alaoglu theorem*

[1.0.1] Definition: The *polar* U^o of an open neighborhood U of 0 in a real or complex topological vector space V is

$$U^o = \{\lambda \in V^* : |\lambda u| \leq 1, \ \text{ for all } u \in U\}$$

[1.0.2] Theorem: *(Banach-Alaoglu)* In the weak star-topology [1] on V^* the polar U^o of an open neighborhood U of 0 in V is *compact.*

Proof: For every v in V there is t_v sufficiently large real such that $v \in t_v \cdot U$. Then $|\lambda v| \leq t_v$ for $\lambda \in U^o$. Let

$$D_v \;=\; \{z \in \mathbb{C} : |z| \leq t_v\} \;\subset\; \mathbb{C}$$

Tychonoff gives compactness of the product

$$P \;=\; \prod_{v\in V} D_v \;\subset\; \prod_{v\in V} \mathbb{C}$$

Map V^* to $\prod_{v\in V} C$ by $j(\lambda) = \{\lambda(v) : v \in V\}$. By design,

$$j(U^o) \;\subset\; P$$

To prove the compactness of U^o it suffices to show that the weak-*-topology on U^o is identical to the subspace topology on $j(U^o)$ inherited from P, and that $j(U^o)$ is closed in P.

Regarding the topologies, the sub-basis sets

$$\{\lambda \in V^* : |\lambda v - \lambda_o v| < \delta\} \qquad\qquad (\text{for } v \in V \text{ and } \delta > 0)$$

for V^* are mapped by j to the sub-basis sets

$$\{p \in P : |p_v - \lambda_o v| < \delta\} \qquad\qquad (\text{for } v \in V \text{ and } \delta > 0)$$

[1] Recall that that weak-*-topology on the dual V^* of a topological vector space V is given by seminorms $p_v(\lambda) = |\lambda(v)|$ for $v \in V$ and $\lambda \in V^*$. Yes, the same expression written as $p_\lambda(v) = |\lambda(v)|$ gives seminorms on V giving it its *weak topology.*

for the product topology on P. That is, j maps U^o with the weak star-topology homeomorphically to $j(U^o)$.

To show that $j(U^o)$ is closed in P, consider L in the closure of U^o in P. Given $x, y \in V$, $a, b \in \mathbb{C}$, the sets

$$\{p \in P : |(p-L)_x| < \delta\} \qquad \{p \in P : |(p-L)_y| < \delta\} \qquad \{p \in P : |(p-L)_{ax+by}| < \delta\}$$

are open in P and contain L, so meet $j(U^o)$. Let $\lambda \in j(U^o)$ lie in the intersection of these three sets and $j(U^o)$. Then

$$|aL_x + bL_y - L_{ax+by}| \;\le\; |a| \cdot |L_x - \lambda x| + |b| \cdot |L_y - \lambda y| + |L_{ax+by} - \lambda(ax+by)| + |a\lambda x + b\lambda y - \lambda(ax+by)|$$

$$\le\; |a| \cdot \delta + |b| \cdot \delta + \delta + 0 \qquad \text{(for every } \delta > 0)$$

so L is linear.

Given $\varepsilon > 0$, let N be a neighborhood of 0 in V such that $x - y \in N$ implies

$$\lambda x - \lambda y \;\in\; N$$

Then

$$|L_x - L_y| \;=\; |L_x - \lambda x| + |L_y - \lambda y| + |\lambda x - \lambda y| \delta + \delta + \varepsilon$$

Thus, L is continuous. Also,

$$|L_x - \lambda x| \;<\; \delta \qquad \text{(for all } x \in U \text{ and all } \delta > 0)$$

so $L \in j(U^o)$, and $j(U^o)$ is closed. ///

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2. *Variant Banach-Steinhaus/uniform boundedness*

This variant of the Banach-Steinhaus (uniform boundedness) theorem is used with Banach-Alaoglu to show that weak boundedness implies boundedness in a locally convex space, the starting point for *weak-to-strong principles*. It uses the version of Baire category for locally compact Hausdorff spaces, rather than complete metric spaces.

[2.0.1] Theorem: *(Variant Banach-Steinhaus)* Let K be a compact convex set in a topological vectorspace X, and $\mathscr{T}$ a set of continuous linear maps $X \to Y$ from X to another topological vectorspace Y. Suppose that for every *individual* $x \in K$ the collection of images

$$\mathscr{T}x \;=\; \{Tx : T \in \mathscr{T}\}$$

is *bounded* in Y. Then $B = \bigcup_{x \in K} \mathscr{T}x$ is bounded in Y.

Proof: Let U, V be balanced neighborhoods of 0 in Y so that $\overline{U} + \overline{U} \subset V$, and let

$$E \;=\; \bigcap_{T \in \mathscr{T}} T^{-1}(\overline{U})$$

By the boundedness of $\mathscr{T}x$, there is a positive integer n such that $\mathscr{T}x \subset nU$, and then $x \in nE$. For every $x \in K$ there is such n, so

$$K \;=\; \bigcup_n (K \cap nE)$$

Since E is closed, the version of the Baire category theorem for locally compact Hausdorff spaces implies that at least one set $K \cap nE$ has non-empty interior in K.

For such n, let x_o be an interior point of $K \cap nE$. Pick a balanced neighborhood W of 0 in X such that

$$K \cap (x_o + W) \ \subset \ nE$$

Since K is compact, it is bounded, so $K - x_o$ is bounded, and for large enough positive real t

$$K \ \subset \ x_o + tW$$

Since K is convex, for any $x \in K$ and $t \geq 1$

$$(1 - t^{-1})x + t^{-1}x \ \in \ K$$

At the same time,

$$z - x_o \ = \ t^{-1}(x - x_o) \in W \qquad \text{(for large enough } t\text{)}$$

by the boundedness of K, so $z \in x_o + W$. Thus,

$$z \in K \cap (x_o + V) \ \subset \ nE$$

From the definition of E, $TE \subset \overline{U}$, so

$$T(nE) \ = \ nT(E) \ \subset \ n\overline{U}$$

And $x = tz - (t-1)x_o$ yields

$$Tx \ \in \ tn\overline{U} - (t-1)n\overline{U} \subset tn(\overline{U} + \overline{U})$$

by the balanced-ness of U. Since $\overline{U} + \overline{U} \subset V$, we have $B \subset tnV$. Since V was arbitrary, this proves the boundedness of B. ///

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3. *Second polars*

The *second polar*, or *bipolar* N^{oo} of an open neighborhood N of 0 in a topological vector space V is

$$N^{oo} \ = \ \{v \in V : |\lambda\, v| \leq 1 \text{ for all } \lambda \in N^o\}$$

where N^o is the polar of N.

[3.0.1] Proposition: *(On second polars)* For V a locally convex topological vectorspace and N a convex, balanced neighborhood of 0, the bipolar N^{oo} of N is the closure $\overline{N}$ of N.

Proof: Certainly N is contained in N^{oo}, and in fact $\overline{N}$ is contained in N^{oo} since N^{oo} is closed. By the local convexity of V, Hahn-Banach implies that for $v \in V$ but $v \notin \overline{N}$ there is $\lambda \in V^*$ such that $\lambda v > 1$ and $|\lambda v'| \leq 1$ for all $v' \in \overline{N}$. Thus, λ is in N^o, and every element $v \in N^{oo}$ is in $\overline{N}$, so $N^{oo} = \overline{N}$. ///

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4. *Weak boundedness implies strong boundedness*

[4.0.1] Theorem: Let V be a locally convex topological vectorspace. A subset E of V is bounded if and only if it is weakly bounded.

Proof: That boundedness implies weak boundedness is trivial. On the other hand, suppose E is weakly bounded, and let U be a neighborhood of 0 in V in the original topology. By local convexity, there is a convex (and balanced) neighborhood N of 0 such that the closure $\overline{N}$ is contained in U.

By the weak boundedness of E, for each $\lambda \in V^*$ there is a bound b_λ such that $|\lambda\, x| \le b_\lambda$ for $x \in E$. By Banach-Alaoglu the polar N^o of N is compact in V^*. The functions $\lambda \to \lambda x$ are continuous, so by variant Banach-Steinhaus there is a uniform constant $b < \infty$ such that $|\lambda x| \le b$ for $x \in E$ and $\lambda \in N^o$. Thus, $b^{-1}x$ is in the bipolar N^{oo} of N, shown by the previous proposition to be the closure $\overline{N}$ of N. That is, $b^{-1}x \in \overline{N}$. By the balanced-ness of N,

$$E \subset t\overline{N} \subset tU \qquad \text{(for any } t > b\text{)}$$

This shows that E is bounded. ///

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5. *Weak-to-strong differentiability*

This is an application of the fact that weak boundedness implies boundedness. The result is well-known folklorically, especially for the case of Banach-space-valued functions, but the simple general case is generally treated as being apocryphal. In fact, weak-versus-strong differentiability and holomorphy were treated definitively in [Grothendieck 1954] and [Grothendieck 1953]. The first of these is cited in [Barros-Neto 1964], for example.

[5.0.1] Definition: Let $f : [a, c] \to V$ be a V-valued function on an interval $[a, c] \subset \mathbb{R}$. The function f is *differentiable* if for each $x_o \in [a, c]$

$$f'(x_o) = \lim_{x \to x_o} (x - x_o)^{-1} \Big(f(x) - f(x_o) \Big)$$

exists. The function f is *continuously differentiable* when it is differentiable and f' is continuous. A k-times continuously differentiable function is said to be C^k, and a continuous function is said to be C^o.

[5.0.2] Definition: A V-valued function f is *weakly* C^k when for every $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ is C^k.

[5.0.3] Remark: The present sense of *weak differentiability* of a function f does *not* refer to distributional derivatives, but to differentiability of every scalar-valued function $\lambda \circ f$ where f is vector-valued and λ ranges over suitable continuous linear functionals. Indeed, nothing like the assertion of the following theorem holds for the *distributional* sense of differentiability: every locally integrable function is distributionally infinitely-differentiable, but rarely continuously-differentiable.

[5.0.4] Theorem: A *weakly* C^k V-valued function f on and interval $[a, c]$, with V a quasi-complete locally convex topological vector space V, is *strongly* C^{k-1}.

Proof: To have f be differentiable at fixed $b \in [a, c]$ is to have *continuity* at b of

$$g(x) \;=\; \frac{f(x) - f(b)}{x - b} \qquad \text{(for } x \ne b\text{)}$$

Weak C^2-ness of f implies that every $\lambda \circ g$ extends to a C^1 scalar-valued function on $[a, c]$. We need to get from this to a continuous extension of g to the whole interval.

The continuity of f' will follow from consideration of the function of two variables (initially for $x \ne y$)

$$g(x, y) \;=\; \frac{f(x) - f(y)}{x - y}$$

The weak C^2-ness of f assures that g extends to a weakly C^1 function on $[a, c] \times [a, c]$. In particular, the function $x \to g(x, x)$ of (the extended) g is weakly C^1. This function is $f'(x)$. Thus, f' is weakly C^1, so is (strongly) C^o.

Suppose that we already know that f is C^ℓ, for $\ell < k-1$. As the ℓ^{th} derivative $g = f^{(\ell)}$ of f is weakly C^2, it is (strongly) C^1 by the first part of the argument. That is, f is $C^{\ell+1}$.

Thus, we need

[5.0.5] Lemma: Let V be a quasi-complete locally convex topological vector space. Fix real numbers $a \le b \le c$. Let g be a V-valued function defined on $[a,b) \cup (b,c]$. Suppose that for $\lambda \in V^*$ the scalar-valued function $\lambda \circ g$ extends to a C^1 function F_λ on the whole interval $[a,c]$. Then $g(b)$ can be chosen such that the extended $g(x)$ is (strongly) continuous on $[a,c]$.

Proof: For each $\lambda \in V^*$, let F_λ be the extension of $\lambda \circ g$ to a C^1 function on $[a,c]$. The differentiability of F_λ implies that for each λ the function

$$\Phi_\lambda(x,y) \;=\; \frac{F_\lambda(x) - F_\lambda(y)}{x-y} \qquad\qquad (\text{for } x \neq y)$$

has a continuous extension $\tilde{\Phi}_\lambda$ to the compact set $[a,c] \times [a,c]$. The image C_λ of $[a,c] \times [a,c]$ under this continuous map is compact in $\mathbb{R}$, so bounded. Thus, the subset

$$\Big\{ \frac{\lambda f(x) - \lambda f(y)}{x-y} : x \neq y \Big\} \;\subset\; C_\lambda$$

is *bounded* in $\mathbb{R}$. That is,

$$E \;=\; \Big\{ \frac{g(x) - g(y)}{x-y} : x \neq y \Big\} \;\subset\; V$$

is *weakly* bounded. Because weakly bounded implies (strongly) bounded, E is (strongly) bounded. That is, for a balanced, convex neighborhood N of 0 in V, there is t_o such that $(g(x)-g(y))/(x-y) \in tN$ for $x \neq y$ in $[a,c]$ and $t \ge t_o$. That is,

$$g(x) - g(y) \;\in\; (x-y)tN$$

Thus, given N and t_o as above,

$$g(x) - g(y) \;\in\; N \qquad\qquad (\text{for } |x-y| < \tfrac{1}{t_o})$$

That is, as $x \to b$ the collection $g(x)$ is a bounded Cauchy net. By quasi-completeness, define $g(b) \in V$ as the limit of the values $g(x)$. For $x \to y$ the values $g(x)$ approach $g(y)$, so this extension of g is continuous on $[a,c]$. ///

This provides the technical result necessary for the proof of the theorem. ///

[Barros-Neto 1964] J. Barros-Neto, *Spaces of vector-valued real analytic functions*, Trans. AMS **112** (1964), 381–391.

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