(*July 2, 2013*)

Distributions supported at 0

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[This document is http://www.math.umn.edu/˜garrett/m/fun/notes_2012-13/09b_distns_at_zero.pdf]

[0.0.1] Theorem: A distribution u with *support* $\{0\}$ is a (finite) linear combination of Dirac's δ and its derivatives.

Recall the notion of *support* of a distribution.

[0.0.2] Definition: The *support* of a distribution u is the *complement* of the *union* of all open sets $U \in \mathbb{R}^n$ such that

$$u(f) \;=\; 0 \qquad\qquad (\text{for } f \in \mathcal{D}_K \text{ with compact } K \subset U)$$

Proof: Since the space $\mathcal{D}$ of test functions on $\mathbb{R}^n$ is $\mathcal{D} = \mathrm{colim}_K \mathcal{D}_K$, it suffices to classify u in $\mathcal{D}_K^*$ with support $\{0\}$.

We have seen that a continuous linear map T from a *limit* of Banach spaces (such as $\mathcal{D}_K$) to a normed space (such as $\mathbb{C}$) factors through a limitand. Thus, there is an *order* $k \geq 0$ such that u factors through

$$C_K^k \;=\; \{f \in C^k(K) :\; f^{(\alpha)} \text{ vanishes on } \partial K \text{ for all } \alpha \text{ with } |\alpha| \leq k\}$$

We need an auxiliary gadget. Fix a smooth compactly-supported function ψ identically 1 on a neighborhood of 0, bounded between 0 and 1, and (necessarily) identically 0 outside some (larger) neighborhood of 0. For $\varepsilon > 0$ let

$$\psi_\varepsilon(x) \;=\; \psi(\varepsilon^{-1}x)$$

Since the support of u is just $\{0\}$, for all $\varepsilon > 0$ and for all $f \in C_c^\infty(\mathbb{R}^n)$ the support of $f - \psi_\varepsilon \cdot f$ does not include 0, so

$$u(\psi_\varepsilon \cdot f) \;=\; u(f)$$

Thus, for some constant C (depending on k and K, but not on f)

$$|\psi_\varepsilon f|_k \;=\; \sup_{x\in K}\sup_{|\alpha|\leq k} |(\psi_\varepsilon f)^{(\alpha)}(x)| \;\leq\; C \cdot \sup_{|i|\leq k}\sup_{x}\sup_{0\leq j\leq i} \varepsilon^{-|j|}\left|\psi^{(j)}(\varepsilon^{-1}x)\, f^{(i-j)}(x)\right|$$

For f vanishing to order k at 0, that is, $f^{(\alpha)}(0) = 0$ for all multi-indices α with $|\alpha| \leq k$, on a fixed neighborhood of 0, by a Taylor-Maclaurin expansion, for some constant C

$$|f(x)| \;\leq\; C \cdot |x|^{k+1}$$

and, generally, for α^{th} derivatives with $|\alpha| \leq k$,

$$|f^{(\alpha)}(x)| \;\leq\; C \cdot |x|^{k+1-|\alpha|}$$

For some constant C

$$|\psi_\varepsilon f|_k \;\leq\; C \cdot \sup_{|i|\leq k}\sup_{0\leq j\leq i} \varepsilon^{-|j|} \cdot \varepsilon^{k+1-|i|+|j|} \;\leq\; C \cdot \varepsilon^{k+1-|i|} \;\leq\; C \cdot \varepsilon^{k+1-k} \;=\; C \cdot \varepsilon$$

Thus, for all $\varepsilon > 0$, for smooth f vanishing to order k at 0,

$$|u(f)| \;=\; |u(\psi_\varepsilon f)| \;\leq\; C \cdot \varepsilon$$

Thus, $u(f) = 0$ for such f.

That is, u is 0 on the intersection of the kernels of δ and its derivatives $\delta^{(\alpha)}$ for $|\alpha| \le k$. Generally,

[0.0.3] Proposition: A continuous linear function $\lambda \in V^*$ vanishing on the intersection of the kernels of a finite collection $\lambda_1, \ldots, \lambda_n$ of continuous linear functionals on V is a linear combination of the λ_i.

Proof: The linear map

$$q : V \longrightarrow \mathbb{C}^n \quad \text{by} \quad v \longrightarrow (\lambda_1 v, \ldots, \lambda_n v)$$

is *continuous* since each λ_i is continuous, and λ factors through q, as $\lambda = L \circ q$ for some linear functional L on $\mathbb{C}^n$. We know all the linear functionals on $\mathbb{C}^n$, namely, L is of the form

$$L(z_1, \ldots, z_n) \;=\; c_1 z_1 + \ldots + c_n z_n \qquad \text{(for some constants } c_i\text{)}$$

Thus,

$$\lambda(v) \;=\; (L \circ q)(v) \;=\; L(\lambda_1 v, \ldots, \lambda_n v) \;=\; c_1 \lambda_1(v) + \ldots + c_n \lambda_n(v)$$

expressing λ as a linear combination of the λ_i. ///

[0.0.4] Remark: The following lemma resolves a potential confusion.

[0.0.5] Lemma: For compact K inside the *complement* of the support of a distribution u,

$$u(f) \;=\; 0 \qquad \text{(for } f \in \mathcal{D}_K\text{)}$$

Proof: This is plausible, but not utterly trivial. Let $\{U_i : i \in I\}$ be open sets such that for compact K' inside any single U_i and $f \in \mathcal{D}_{K'}$ we have $u(f) = 0$. Let $\{\psi_i : i \in I\}$ be a smooth locally finite *partition of unity*[1] subordinate to $\{U_i : i \in I\}$. Take $f \in \mathcal{D}_{K'}$ for K' compact inside $U = \bigcup_i U_i$. Then

$$f = f \cdot 1 = \sum_i f \cdot \psi_i$$

and the sum is *finite*. Then

$$u(f) = u(\sum_i f \cdot \psi_i) = \sum_i u(f \cdot \psi_i) = \sum_i 0 = 0$$

(The fact that the sum is finite allows interchange of summation and evaluation.) ///

[1] That is, the functions ψ_i are smooth, take values between 0 and 1, sum to 1 at all points, and on any compact there are only finitely-many which are non-zero. The existence of such partitions of unity is not completely trivial to prove.