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Meromorphic continuations of distributions

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We take for granted results about Gelfand-Pettis integrals and the Schwartz-Grothendieck ideas on holomorphic functions with values in quasi-complete locally-convex spaces, such as spaces of tempered distributions with the weak dual topology.

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1. $u_s(x) = |x|^s$

[1.1] Differentiation identity For $\mathrm{Re}(s) \ge 2$, the function $u_s(x) = |x|^s$ is twice-continuously-differentiable. In particular, with the usual Euclidean Laplacian Δ,

$$\Delta|x|^s \;=\; \sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{s}{2}\cdot 2x_i\cdot(|x|^2)^{\frac{s}{2}-1} \;=\; \sum_{i=1}^n \Big(\frac{s}{2}\cdot 2\cdot(|x|^2)^{\frac{s}{2}-1} + \frac{s}{2}(\frac{s}{2}-1)\cdot(2x_i)^2\cdot(|x|^2)^{\frac{s}{2}-2}\Big)$$

$$=\; ns\cdot|x|^{s-2} + s(s-2)|x|^{s-2} \;=\; s(s+n-2)\cdot|x|^{s-2}$$

We see that $n = 2$ is anomalous, because the two linear factors become identical, and we ignore this case.

[1.2] Meromorphic continuation The identity $\Delta u_s = s(s+n-2)\cdot u_{s-2}$ at first holds for $\mathrm{Re}(s) \ge 2$, as an equality of continuous functions. At the same time, u_s analytically continues as an $L^1_{\mathrm{loc}}(\mathbb{R}^n)$-valued function of s, therefore as a tempered distribution-valued function of s, to $\mathrm{Re}(s) > -n$. Thus, Δu_s exists as tempered distribution at least on $\mathrm{Re}(s) > -n$. Rewrite the identity as

$$u_{s-2} \;=\; \frac{\Delta u_s}{s(s+n-2)}$$

and replace s by $s+2$:

$$u_s \;=\; \frac{\Delta u_{s+2}}{(s+2)(s+n)}$$

This expression makes sense of u_s as tempered distribution on $\mathrm{Re}(s) > -n-2$ except for possible poles at $s = -n$ and $s = -2$. For $n > 2$, in fact there is no pole at $s = -2$, because u_{-2} is locally integrable. Indeed, $\Delta u_0 = 0$, so $\Delta u_{s+2}/(s+2)$ is holomorphic at $s = -2$.

Repeat:

$$u_s \;=\; \frac{\Delta u_{s+2}/(s+2)}{s+n} \;=\; \frac{\Delta^2 u_{s+4}/(s+2)(s+4)}{(s+n)(s+n-2)}$$

The factors $(s+2)(s+4)$ are indeed cancelled by the vanishing of $\Delta^2 u_2$ and $\Delta^2 u_4$, leaving possible poles at $s = -n, -n-2$. Continuing, u_s extends to a meromorphic tempered-distribution-valued function on $\mathbb{C}$, with poles at most at $s = -n, -n-2, -n-4, \ldots$

[1.3] Regularization and $\mathrm{Res}_{s=-n} u_s = \delta \times$ const With $n \ne 2$, the *first* (rightmost) pole of u_s, at $s = -n$, is a multiple of Dirac δ at 0, seen as follows. Indeed, *locally* away from $x = 0$, we have the vanishing $\Delta u_{2-n}(x) = 0$, showing that the support of Δu_{2-n} is $\{0\}$, as expected.

With Gaussian $\gamma(x) = e^{-|x|^2}$, given Schwartz function f, the difference $f - f(0)\cdot\gamma$ vanishes at 0, so the integral for

$$u_s\big(f(x) - f(0)\cdot\gamma(x)\big) \;=\; \int_{\mathbb{R}^n} |x|^s\cdot\big(f(x) - f(0)\cdot\gamma(x)\big)\,dx$$

is absolutely convergent for $\mathrm{Re}(s) > -n-1$, and is a holomorphic function of s in that half-plane. The identity principle assures that this analytic continuation correctly evaluates u_s on $f - f(0)\cdot\gamma$. In particular, there is no pole at $s = -n$. Thus,

$$\big(\mathrm{Res}_{s=-n} u_s\big)(f) \;=\; \big(\mathrm{Res}_{s=-n} u_s\big)(f - f(0)\cdot\gamma) + f(0)\cdot\big(\mathrm{Res}_{s=-n} u_s\big)(\gamma) \;=\; 0 + f(0)\cdot\big(\mathrm{Res}_{s=-n} u_s\big)(\gamma)$$

Since $f(0) = \delta(f)$, the residue is a constant multiple of δ, with constant

$$\mathrm{Res}_{s=-n}\int_{\mathbb{R}^n} |x|^s\, e^{-|x|^2}\, dx \;=\; |S^{n-1}|\cdot \mathrm{Res}_{s=-n}\int_0^\infty t^s\, e^{-t^2}\, t^{n-1}\, dt \;=\; |S^{n-1}|\cdot \tfrac{1}{2}\mathrm{Res}_{s=-n}\int_0^\infty t^{\frac{s+n}{2}}\, e^{-t}\, \frac{dt}{t}$$

$$=\; |S^{n-1}|\cdot \tfrac{1}{2}\mathrm{Res}_{s=-n}\int_0^\infty t^{\frac{s+n}{2}}\, e^{-t}\, \frac{dt}{t} \;=\; |S^{n-1}|\cdot\tfrac{1}{2}\mathrm{Res}_{s=-n}\cdot\Gamma(\frac{s+n}{2})$$

$$=\; |S^{n-1}|\cdot\tfrac{1}{2}\mathrm{Res}_{s=-n}\frac{2}{s+n} \;=\; |S^{n-1}| \;=\; \text{natural measure of } S^{n-1}$$

[1.4] Solving $\Delta u = \delta$ The distribution-valued function $(s+n)u_s$ takes value $\mathrm{Res}_{s=-n} u_s$ at $s=-n$. By the identity principle, the equality

$$\Delta u_{s+2} \;=\; (s+2)\cdot(s+n)u_s$$

also holds at $s = -n$, so

$$\Delta\,\frac{1}{|x|^{n-2}} \;=\; \Delta u_{-n+2} \;=\; (-n+2)\cdot|S^{n-1}|\cdot\delta \qquad\qquad \text{(distributionally)}$$

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2. *Rational Dirac comb* $u_s = \sum_{\frac{p}{q}} \frac{1}{q^s}\cdot\delta_{p/q}$

The usual *Dirac comb* is

$$\text{Dirac comb} \;=\; \sum_{n\in\mathbb{Z}} \delta_n$$

A slightly more complicated comb consisting of a weighted linear combination of Dirac δ at rational numbers is

$$u_s \;=\; \sum_{0<\frac{p}{q}\le 1} \frac{1}{q^s}\cdot\delta_{p/q} \qquad\qquad \text{(fraction } p/q \text{ in lowest terms)}$$

viewed as on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For $\mathrm{Re}(s) > 2$ this gives a distribution on $\mathbb{T}$.

[2.1] Rewriting without lowest-terms condition As often happens, the *fraction-in-lowest-terms* condition can be understood in terms of a similar object without the condition: noting that $\delta_{pd/qd} = \delta_{p/q}$,

$$\zeta(s)\cdot u_s \;=\; \sum_{d\ge 1}\frac{1}{d^s}\cdot\sum_{q\ge 1}\Big(\frac{1}{q^s}\sum_{0<p\le q,\ \gcd(p,q)=1}\delta_{p/q}\Big) \;=\; \sum_{d\ge 1}\sum_{q\ge 1}\frac{1}{(qd)^s}\sum_{0<pd\le qd,\ \gcd(pd,qd)=d}\delta_{pd/qd}$$

Replacing qd by q and pd by p, this is

$$=\; \sum_{q=1}^\infty \frac{1}{q^s}\sum_{d|q}\ \sum_{0<p\le q,\ \gcd(p,q)=d}\delta_{p/q} \;=\; \sum_{q=1}^\infty\frac{1}{q^s}\sum_{0<p\le q}\delta_{p/q}$$

Denote the latter by v_s, so $\zeta(s)\cdot u_s = v_s$.

[2.2] Fourier expansion For $\mathrm{Re}(s) > 2$, the n^{th} Fourier coefficient of v_s is

$$\widehat{v}_s(n) \;=\; v_s(e^{-2\pi inx}) \;=\; \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{0<p\le q} \delta_{p/q}(e^{-2\pi inx}) \;=\; \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{0<p\le q} e^{-2\pi in\frac{p}{q}}$$

The function $p \to e^{-2\pi in\frac{p}{q}}$ is a character on $\mathbb{Z}/q$, non-trivial unless $q|n$, so the sum over p is 0 unless $q|n$, in which case it is q. Thus,

$$\widehat{v}_s(n) \;=\; \sum_{q\ge 1,\; q|n} \frac{1}{q^{s-1}}$$

Denoting the sum of α^{th} powers of positive divisors of n by $\sigma_\alpha(n)$,

$$\widehat{v}_s(n) \;=\; \begin{cases} \sigma_{1-s}(n) & (\text{for } n \neq 0) \\ \zeta(s-1) & (\text{for } n = 0) \end{cases}$$

and

$$\widehat{u}_s(n) \;=\; \begin{cases} \dfrac{\sigma_{1-s}(n)}{\zeta(s)} & (\text{for } n \neq 0) \\ \dfrac{\zeta(s-1)}{\zeta(s)} & (\text{for } n = 0) \end{cases}$$

Even the crudest estimate $\sigma_\alpha(n) \le |n| \cdot |n|^\alpha$ demonstrates the polynomial grown of coefficients of $\widehat{u}_s$, so u_s meromorphically continues as a distribution on $\mathbb{T}$.

The pole of $\zeta(s)$ at $s = 1$, makes all Fourier coefficients of u_s vanish at $s = 1$, since the functional equation of $\zeta(s)$ gives

$$\zeta(0) \;=\; \frac{\pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s)}{\pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})}\Big|_{s=0} \;=\; \frac{\pi^{-\frac{1-s}{2}}\,\Gamma(\frac{1-s}{2})\,\zeta(1-s)}{\pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})}\Big|_{s=0} \;=\; \pi^{-\frac{1}{2}} \cdot \Gamma(\tfrac{1}{2}) \cdot \frac{\zeta(1-s)}{\Gamma(\frac{s}{2})}\Big|_{s=0}$$

$$=\; \frac{\dfrac{1}{(1-s)-1} + (\text{holomorphic at } s=0)}{\dfrac{2}{s} + (\text{holomorphic at } s=0)}\Big|_{s=0} \;=\; -\tfrac{1}{2}$$

That is, after analytic continuation,

$$u_1 \;=\; 0$$
