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Simplest case of Fredholm alternative

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Here, we prove the simplest *Fredholm alternative*:

[1.1] Theorem: For Hilbert space X, for compact $T : X \longrightarrow X$ and $0 \neq \lambda \in \mathbb{C}$, $T - \lambda$ has closed image of codimension equal to the dimension of its kernel. *(Proof in the sequel.)*

[1.2] Corollary: For $\lambda \neq 0$, *either* $T - \lambda$ is a bijection, *or* λ is an eigenvalue. ///

[1.3] Corollary: The only non-zero spectrum of a compact operator is *point* spectrum. [1] ///

[1.4] Corollary: *Either* λ is an eigenvalue, *or* $(T - \lambda)u = v$ is solvable for u for all $v \in X$. ///

This result also complements the spectral theorem for *self-adjoint* compact operators on a Hilbert space, where there is an orthonormal basis of eigenvectors. For not-necessarily-self-adjoint (or not-necessarily-normal) compact operators, it can happen that there are *no* non-zero eigenvalues. This is not a pathology: the Volterra operator

$$Vf(x) \;=\; \int_0^x f(y)\, dy \qquad\qquad (\text{for } f \in L^2[0,1])$$

is Hilbert-Schmidt, hence compact, but has no non-zero eigenvalues.

[1.5] Compact operators invertible only on finite-dimensional

For compact $T : X \to X$ with continuous inverse T^{-1}, the boundedness of T^{-1} gives a constant C such that $|T^{-1}x| \leq C \cdot |x|$ for all $x \in Y$. Invertibility implies that $TX = X$, and $|x| \leq C \cdot |Tx|$ for all $x \in X$. Thus, the image by T of the unit ball in X contains an open ball in X. Compactness implies that X is finite-dimensional.

[1.6] Generalized eigenspaces finite-dimensional for $\lambda \neq 0$

For compact $T : X \to X$ and $\lambda \neq 0$, the kernel of $T - \lambda$ is finite-dimensional, since any restriction of T to a subspace is still compact, and T acts by a scalar on $\ker(T - \lambda)$.

By induction on n, the operator $T - \lambda$ maps $\ker(T - \lambda)^{n+1}$ to the finite-dimensional space $\ker(T - \lambda)^n$, so is finite-rank. On $\ker(T - \lambda)^{n-1}$,

$$\text{compact} \;=\; \text{finite-rank} \;=\; T - \lambda \;=\; \text{compact} - \lambda \qquad\qquad (\text{on } \ker(T - \lambda)^{n+1})$$

Thus, $\lambda \neq 0$ is compact on $\ker(T - \lambda)^{n+1}$, implying that this kernel is finite-dimensional. ///

[1.7] T compact if and only if T^* compact

Proof: First, the adjoint map $T \to T^*$ is continuous in the operator-norm topology. Indeed, $|T^*| = |T|$, because

$$|T^*|^2 \;=\; \sup_{|x| \leq 1} |T^*x|^2 \;=\; \langle T^*x, T^*x \rangle \;=\; \langle TT^*x, x \rangle \;\leq\; |TT^*x| \cdot |x| \;\leq\; |T| \cdot |T^*| \cdot |x| \cdot |x| \;=\; |T| \cdot |T^*|$$

[1] Recall that the *eigenvalues* or *point spectrum* of an operator T on a Hilbert space X consists of $\lambda \in \mathbb{C}$ such that $T - \lambda$ fails to be *injective*. The *continuous* spectrum consists of λ with $T - \lambda$ *injective* and with *dense* image, but *not surjective*. The *residual spectrum* consists of λ with $T - \lambda$ *injective* but $(T - \lambda)X$ not *dense*.

Dividing through by $|T^*|$ gives $|T^*| \le |T|$. Symmetrically, $|T| \le |T^*|$. Compact T is an operator-norm limit of finite-rank operators T_n. Then T^* is the operator-norm limit of the finite-rank T_n^*. ///

[1.8] $\mathrm{Im}(T-\lambda)$ is closed for $\lambda \neq 0$

Proof: Let $(T-\lambda)x_n \to y$. First consider the situation that $\{x_n\}$ is *bounded*. Compactness of T yields a convergent subsequence of Tx_n, and we replace x_n by the corresponding subsequence. Then $-\lambda x_n = y - Tx_n$ converges to $y - \lim Tx_n$, so x_n is convergent to $x_o \in X$, since $\lambda \neq 0$, and $Tx_o = y$.

To reduce the general case to the previous, first reduce to the case that $T-\lambda$ is *injective*: from above, $\ker(T-\lambda)$ is finite-dimensional. Let V be the orthogonal complement to $\ker(T-\lambda)$. Since $(T-\lambda)V = (T-\lambda)X$, to prove the image is closed it suffices to consider V, or, equivalently, that $T-\lambda$ is *injective* on X. Since $T-\lambda$ is a continuous bijection to its image, by the *open mapping theorem* it is an isomorphism to its image. Thus, there is $\delta > 0$ such that $|(T-\lambda)x| \ge \delta|x|$.

Returning to the main argument, suppose that $(T-\lambda)x_n \to y_o$. Then $(T-\lambda)(x_m - x_n) \to 0$. For $T-\lambda$ injective, $x_m - x_n \to 0$, so x_n is *bounded*, reducing to that case. ///

[1.9] $T-\lambda$ injective $\iff$ surjective for $\lambda \neq 0$

Proof: Suppose $T-\lambda$ is injective. Let $V_n = (T-\lambda)^n X$. Since images under $T-\lambda$ for compact T and $\lambda \neq 0$ are closed, by induction these are *closed* subspaces of X. For $x \notin (T-\lambda)X$ and any $y \in X$,

$$(T-\lambda)^n x - (T-\lambda)^{n+1} y \;=\; (T-\lambda)^n \Big(x - (T-\lambda)y \Big)$$

Injectivity of $T-\lambda$ implies that of $(T-\lambda)^n$, so this is not 0. That is, $(T-\lambda)^n x \notin (T-\lambda)^{n+1}X$. Thus, the chain of subspaces V_n is strictly decreasing.

Take $v_n \in V_n$ such that $|v_n| = 1$ and away from V_n, say by

$$\inf_{y \in V_{n+1}} |v_n - y| \;\ge\; \tfrac{1}{2}$$

The effect of T is

$$Tv_m - Tv_{m+n} \;=\; \lambda v_m + (T-\lambda)v_m - Tv_{m+n} \;\in\; \lambda v_m + V_{n+1} \qquad \text{(integers } m \ge 1 \text{ and } n \ge 1)$$

since V_{m+1} is T-stable. Thus,

$$|Tv_m - Tv_{m+n}| \;\ge\; |\lambda| \cdot \tfrac{1}{2}$$

This is impossible, since compact T maps the bounded set $\{v_n\}$ to a pre-compact set. Thus, the chain of subspaces V_n cannot be strictly decreasing, and have surjectivity $(T-\lambda)X = X$.

On the other hand, suppose $T-\lambda$ is *surjective*. Then the adjoint $(T-\lambda)^*$ is *injective*. Since adjoints of compact operators are compact, we already know that $(T-\lambda)^*$ is *surjective*. Then $T-\lambda = (T-\lambda)^{**}$ is *injective*. ///

[1.10] $\dim\ker(T-\lambda) = \dim\mathrm{coker}(T-\lambda)$ for $\lambda \neq 0$, T compact

That is, such operators are *Fredholm operators of index* 0.

Proof: The compactness of T entails the finite-dimensionality of $\ker(T-\lambda)$ for $\lambda \neq 0$. Dually, for $y_1, \ldots, y_n \in X$ linearly independent modulo $(T-\lambda)X$, by *Hahn-Banach* there are $\eta_1, \ldots, \eta_n \in X^*$ vanishing on the image $(T-\lambda)X$ and $\eta_i(y_j) = \delta_{ij}$. Such η_i are in the kernel of the adjoint $(T-\lambda)^*$. We know T^* is compact, so $\ker(T-\lambda)^*$ is finite-dimensional.

We've proven that injectivity and surjectivity of $T - \lambda$ are equivalent, and that the kernel and cokernel are finite-dimensional. Let $x_1, \ldots, x_m$ (with $m \geq 1$) span the kernel, and let (the images of) $y_1, \ldots, y_n$ (with $n \geq 1$) span the cokernel, and show that $m = n$.

For $m \leq n$, let X' be a closed complementary subspace to the kernel of $T - \lambda$, for example, its orthogonal complement. Let F be the finite-rank operator which is 0 on X' and $Fx_i = y_i$. The adjusted operator $T' = T + F$ is compact. For $(T' - \lambda)x = 0$,

$$(T - \lambda)x \;=\; Fx \;\in\; (T - \lambda)X \cap \operatorname{span} y_1, \ldots, y_n \;=\; \{0\}$$

That is, $T' - \lambda$ is *injective*, so is *surjective*, so $m = n$. In the opposite case $m \geq n$, let $Fx_i = y_i$ for $i \leq n$, and $Fx_i = y_n$ for $i \geq n$. With $T' = T + F$ again, in this case $T' - \lambda$ is *surjective*, so is injective, and $m = n$. ///

[1.11] Discreteness of spectrum of compact operators

[1.12] Claim: For T a compact operator on a Hilbert the non-zero spectrum (if any) is *point* spectrum. The number of eigenvalues λ outside a given disk $|\lambda| \leq r$ is *finite* for $r > 0$, and always 0 is in the spectrum.

Proof: For λ not an eigenvalue, we know that $T - \lambda$ is injective *and surjective*, so by the open mapping theorem it is an isomorphism. Thus, indeed, the only non-zero spectrum consists of *eigenvalues*. We also know that eigenspaces are finite-dimensional, for non-zero eigenvalues.

For infinite-dimensional Hilbert spaces, 0 inevitably lies in the spectrum, otherwise T would be *invertible*. Then $1 = T \circ T^{-1}$ is the composition of a compact operator and a continuous operator, so is compact, which is possible only in finite-dimensional spaces.

Suppose there were infinitely-many different eigenvalues $\lambda_1, \lambda_2, \ldots$ outside the closed disk $|\lambda| \leq r$ with $r > 0$, with corresponding eigenvectors x_i with $|x_i| = 1$. First, the x_i are linearly independent: let $\sum_i c_i\, x_i = 0$ be a non-trivial linear dependence relation with fewest non-zero c_i's, and apply T: for an index i_o with $c_{i_o} \neq 0$, we obtain a shorter relation by suitable subtraction,

$$0 \;=\; \sum_i \lambda_i\, c_i\, x_i - \lambda_{i_o} \sum_i c_i\, x_i \;=\; \sum_{i \neq i_o} (\lambda_i - \lambda_{i_o})\, c_i\, x_i$$

Thus, there can be no non-trivial linear dependence. With V_n the span of $x_1, x_2, \ldots, x_n$, this implies that the containments $V_n \subset V_{n+1}$ are *strict*. Thus, there exist unit vectors $y_i \in V_i$ with the distance from y_i to V_{i-1} at least $\frac{1}{2}$. Then for $i > j$

$$Ty_i - Ty_j \;=\; \lambda_i y_i + (T - \lambda_i)y_i - Ty_j \;\in\; \lambda_i y_i + V_{i+1}$$

and, thus, $|Ty_i - Ty_j| \geq |\lambda| \cdot \frac{1}{2}$. However, this contradicts the compactness of T. We conclude that there can be only finitely-many eigenvalues larger than $r > 0$. ///

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