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The Hilbert transform

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Formulaically, the Cauchy principal-value functional η is

$$\eta f \;=\; \text{principal-value functional of } f \;=\; P.V.\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{f(x)}{x}\,dx$$

This is a somewhat fragile presentation. In particular, the apparent integral is *not* a literal integral!

The uniqueness proven below helps prove plausible properties like the *Sokhotski-Plemelj theorem* from [Sokhotski 1871], [Plemelji 1908], with a possibly unexpected leading term:

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\,dx \;=\; -i\pi\, f(0) + P.V.\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx$$

See also [Gelfand-Silov 1964]. Other plausible identities are best certified using uniqueness, such as

$$\eta f \;=\; \tfrac{1}{2} \int_{\mathbb{R}} \frac{f(x)-f(-x)}{x}\,dx \qquad\qquad (\text{for } f \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}))$$

using the canonical continuous extension of $\frac{f(x)-f(-x)}{x}$ at 0. Also,

$$\eta f \;=\; \int_{\mathbb{R}} \frac{f(x)-f(0)\cdot e^{-x^2}}{x}\,dx \qquad\qquad (\text{for } f \in C^o(\mathbb{R}) \cap L^1(\mathbb{R}))$$

The *Hilbert transform* of a function f on $\mathbb{R}$ is awkwardly described as a principal-value integral

$$(\mathcal{H}f)(x) \;=\; \frac{1}{\pi} P.V.\int_{-\infty}^{\infty} \frac{f(t)}{x-t}\,dt \;=\; \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \int_{|t-x|>\varepsilon} \frac{f(t)}{x-t}\,dt$$

with the leading constant $1/\pi$ understandable with sufficient hindsight: we will see that this adjustment makes $\mathcal{H}$ extend to a *unitary* operator on $L^2(\mathbb{R})$. The formulaic presentation of $\mathcal{H}$ makes it appear to be a *convolution* with (a constant multiple of) the principal-value functional. The fragility of these presentations make existence, continuity properties, and range of applicability of $\mathcal{H}$ less clear than one would like.

1. *The principal-value functional*

The principal-value functional η is better characterized as the unique (up to a constant multiple) odd *distribution* on $\mathbb{R}$, positive-homogeneous of degree 0 as a distribution (see below). This characterization allows unambiguous comparison of various limiting expressions closely related to the principal-value functional. Further, the characterization of the principal-value functional makes discussion of the Hilbert transform less fragile and more convincing.

Again, let

$$\eta f \;=\; \text{principal-value functional of } f \;=\; P.V.\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|>\varepsilon} \frac{f(x)}{x}\,dx$$

We prove that η is a tempered distribution:

[1.1] Claim: For $f\in\mathscr{S}$, for every $h>0$,

$$|\eta(f)| \;\le\; \int_{|x|\ge h} \Big|\frac{f(x)}{x}\Big|\,dx + 2h \sup_{|x|\le h} |f'(x)|$$

Thus, u is a tempered distribution:

$$|\eta(f)| \;\ll\; \sup_x (1+x^2)|f(x)| + \sup_x |f'(x)| \qquad\qquad \text{(implied constant independent of } f)$$

Proof: Certainly

$$|\eta(f)| \;=\; \int_{|x|\ge h} \Big|\frac{f(x)}{x}\Big|\,dx + \lim_{\varepsilon\to 0^+} \Big|\int_{\varepsilon<|x|\le h} \frac{f(x)}{x}\,dx\Big|$$

By the Mean Value Theorem, $f(x)=f(0)+x\,f'(\xi_x)$ for some ξ_x between 0 and x. Thus,

$$\int_{\varepsilon<|x|\le h} \frac{f(x)}{x}\,dx \;=\; \int_{\varepsilon<|x|\le h} \frac{f(0)}{x}\,dx + \int_{\varepsilon<|x|\le h} f(\xi x)\,dx \;=\; 0 + \int_{\varepsilon<|x|\le h} f(\xi x)\,dx$$

because $1/x$ is *odd* and $x\to f(0)$ is *even*. Thus,

$$\Big|\int_{\varepsilon<|x|\le h} \frac{f(x)}{x}\,dx\Big| \;=\; \Big|\int_{\varepsilon<|x|\le h} f(\xi x)\,dx\Big| \;\le\; \int_{\varepsilon<|x|\le h} |f(\xi x)|\,dx$$

$$\le\; 2h\cdot \sup_{\varepsilon<|x|\le h} |f'(x)| \;\le\; 2h\cdot \sup_{|x|\le h} |f'(x)|$$

The latter is independent of ε. With $0<h\le 1$,

$$\int_{|x|\ge h} \Big|\frac{f(x)}{x}\Big|\,dx \;=\; \int_{|x|\ge 1} \Big|\frac{f(x)}{x}\Big|\,dx + \int_{h\le|x|\le 1} \Big|\frac{f(x)}{x}\Big|\,dx$$

$$\le\; \int_{|x|\ge 1} |f(x)|\,dx + \frac{1}{h}\int_{h\le|x|\le 1} |f(x)|\,dx \;\le\; \int_{|x|\ge 1} (1+x^2)|f(x)|\cdot\frac{1}{1+x^2}\,dx + 2h\sup_x |f(x)|$$

$$\le\; \sup_x (1+x^2)|f(x)|\cdot \int_{|x|\ge 1} \frac{1}{1+x^2}\,dx + 2h\sup_x |f(x)| \;\ll\; \sup_x (1+x^2)|f(x)|$$

Of course, $\sup_{|x|\le h} |f'(x)| \le \sup_x |f'(x)|$, giving the corollary. ///

To make the *degree* of a positive-homogeneous function φ agree with the degree of the integration-against-φ distribution u_φ, due to change-of-measure, we find that

$$(u_\varphi \circ t)(f) \;=\; u_\varphi(f \circ t^{-1}) \;=\; \int_{\mathbb{R}} u_\varphi(x)\, f(t^{-1}x)\, dx \;=\; \int_{\mathbb{R}} u_\varphi(tx)\, f(x)\, d(tx) \;=\; t \cdot u_{\varphi \circ t}(f)$$

Thus, for agreement of the notion of homogeneity for distributions and (integrate-against) functions, the dilation action $u \to u \circ t$ on *distributions* should be

$$(u \circ t)(f) \;=\; \frac{1}{t} u(f \circ t^{-1}) \qquad \text{(for } t > 0\text{, test function } f\text{, and distribution } u\text{)}$$

Parity is as expected: u is *odd* when $u(x \to f(-x)) = -u(x \to f(x))$ for all test functions f, and is *even* when $u(x \to f(-x)) = +u(x \to f(x))$ for all test functions f.

[1.2] Claim: The principal-value distribution η is positive-homogeneous of degree -1, and is *odd*.

Proof: The ε^{th} integral in the limit definition of η is itself *odd*, by changing variables:

$$\eta(f \circ (-1)) \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(-x)}{x}\, dx \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(x)}{-x}\, d(-x)$$

$$=\; \lim_{\varepsilon \to 0^+} -\int_{|x| \geq \varepsilon} \frac{f(x)}{x}\, dx \;=\; -\lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(x)}{x}\, dx \;=\; -\eta(f)$$

as claimed. For the degree of homogeneity, for $t > 0$ and test function f,

$$(\eta \circ t)(f) \;=\; \frac{1}{t}\eta(f \circ \frac{1}{t}) \;=\; \frac{1}{t} \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(\frac{x}{t})}{x}\, dx \;=\; \frac{1}{t} \lim_{\varepsilon \to 0^+} \int_{|tx| \geq \varepsilon} \frac{f(x)}{tx}\, d(tx) \;=\; \frac{1}{t} \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon/t} \frac{f(x)}{x}\, d(x)$$

$$=\; \frac{1}{t} \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(x)}{x}\, d(x) \;=\; \frac{1}{t}\eta(f)$$

That is, η is homogeneous of degree -1. ///

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2. *Other descriptions of the principal-value functional*

[2.1] Claim: Let φ be any *even* function in $L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, with $\varphi(0) = 1$. Then

$$P.V. \int_{\mathbb{R}} \frac{f(x)}{x}\, dx \;=\; \int_{\mathbb{R}} \frac{f(x) - f(0) \cdot \varphi(x)}{x}\, dx \qquad \text{(for } f \in C^o(\mathbb{R}) \cap L^1(\mathbb{R})\text{, for example)}$$

where $\big(f(x) - f(0) \cdot \varphi(x)\big)/x$ is extended by continuity at $x = 0$.

Proof: By the demonstrated odd-ness of the principal-value integral, it is 0 on the even function φ. Extending $\frac{f(x) - f(0) \cdot \varphi(x)}{x}$ by continuity at 0,

$$\int_{\mathbb{R}} \frac{f(x) - f(0) \cdot \varphi(x)}{x}\, dx \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(x) - f(0) \cdot \varphi(x)}{x}\, dx$$

$$=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(x)}{x}\, dx + f(0) \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x}\, dx \;=\; P.V. \int_{\mathbb{R}} \frac{f(x)}{x}\, dx + 0$$

as claimed. ///

The leading term in the following might be unexpected:

[2.2] Claim: *(Sokhotski-Plemelj)*

$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} \frac{f(x)}{x+i\varepsilon}\,dx \;=\; -i\pi\, f(0) + P.V.\int_{-\infty}^{\infty} \frac{f(x)}{x}\,dx$$

Proof: Aiming to apply the previous claim,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{f(x)}{x+i\varepsilon}\,dx \;=\; \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{f(x) - \frac{f(0)}{1+x^2}}{x+i\varepsilon}\,dx + f(0)\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{1/(1+x^2)}{x+i\varepsilon}\,dx$$

For $0<\varepsilon<1$, the right-most integral is evaluated by residues, moving the contour through the upper half-plane:

$$\int_{\mathbb{R}} \frac{1/(1+x^2)}{x+i\varepsilon}\,dx \;=\; 2\pi i\,\mathrm{Res}_{x=i}\frac{1}{(1+x^2)(x+i\varepsilon)} \;=\; 2\pi i \frac{1}{(i+i)(i+i\varepsilon)} \;\longrightarrow\; \frac{\pi}{i} \;=\; -i\pi$$

as claimed. ///

[2.3] Claim: With f continuously differentiable at 0, thereby extending $\frac{f(x)-f(-x)}{x}$ by continuity at 0,

$$\tfrac{1}{2}\int_{\mathbb{R}} \frac{f(x)-f(-x)}{x}\,dx \;=\; P.V.\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx \qquad\qquad (\text{for } f\in C^1(\mathbb{R})\cap L^1(\mathbb{R}))$$

Proof: For test function f, since $\frac{f(x)-f(-x)}{x}$ is continuous as 0, using the odd-ness of the principal-value functional,

$$\tfrac{1}{2}\int_{\mathbb{R}} \frac{f(x)-f(-x)}{x}\,dx \;=\; \tfrac{1}{2}\lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon} \frac{f(x)-f(-x)}{x}\,dx \;=\; \lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{f(x)}{x}\,dx$$

as claimed. ///

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3. *Uniqueness of odd/even homogeneous distributions*

After some clarification of notions of parity and positive-homogeneity, we show that there is a unique distribution of a given parity and positive-homogeneity degree.

[3.1] Claim: For a positive-homogeneous distribution u of degree s, $x\frac{d}{dx}u$ is again positive-homogeneous of degree s, and with the same parity.

Proof: For test function f,

$$((x\frac{d}{dx}u)\circ t)(f) \;=\; \frac{1}{t}(x\frac{d}{dx}u)(f\circ t^{-1}) \;=\; \frac{1}{t}(\frac{d}{dx}u)(x\cdot f\circ t^{-1}) \;=\; -\frac{1}{t}u(\frac{d}{dx}(x\cdot(f\circ t^{-1})))$$

$$=\; -\frac{1}{t}u(f\circ t^{-1} + x\cdot t^{-1}\cdot(f'\circ t^{-1})) \;=\; -\frac{1}{t}u(f\circ t^{-1} + (x\cdot f')\circ t^{-1})) \;=\; -\frac{1}{t}u((f+xf')\circ t^{-1})$$

$$=\; -(u\circ t)(f+xf') \;=\; -(u\circ t)(\frac{d}{dx}(xf)) \;=\; t^s\cdot\big(-u(\frac{d}{dx}(xf))\big) \;=\; t^s\cdot(x\frac{d}{dx}u)(f)$$

as asserted. Preservation of parity is similar: writing $f^-(x) = f(-x)$ for functions f, and $u^-(f) = u(f^-)$ for distributions u,

$$(x\frac{d}{dx}u)^-(f) \;=\; (x\frac{d}{dx}u)(f^-) \;=\; (\frac{d}{dx}u)(x\cdot f^-) \;=\; -u(\frac{d}{dx}(x\cdot f^-)) \;=\; u(\frac{d}{dx}((x\cdot f)^-))$$

$$= \; u(-(\frac{d}{dx}(x\cdot f))^-) \;=\; -u^-(\frac{d}{dx}(x\cdot f)) \;=\; ((x\frac{d}{dx})u^-)(f)$$

as claimed. ///

The *Euler operator* $x\frac{d}{dx}$ is the infinitesimal form of *dilation* $f(x) \to f(tx)$. This is manifest in the following:

[3.2] Corollary: $x\frac{d}{dx}u = s\cdot u$ for positive-homogeneous u of degree s.

Proof: Differentiate the distribution-valued equality $u\circ t = t^s\cdot u$ with respect to t and set $t=1$: the right-hand side is $s\cdot u$. For the left-hand side, for test function f, since $t\to u\circ t$ is a smooth distribution-valued function of t,

$$(\frac{\partial}{\partial t}(u\circ t))(f) \;=\; \frac{\partial}{\partial t}((u\circ t)(f)) \;=\; \frac{\partial}{\partial t}(\frac{1}{t}u(f\circ\frac{1}{t})) \;=\; u(\frac{\partial}{\partial t}(\frac{1}{t}f\circ\frac{1}{t})) \;=\; u\Big(\frac{-1}{t^2}(f\circ\frac{1}{t}) - \frac{x}{t^2}\cdot(f'\circ\frac{1}{t})\Big)$$

Evaluating at $t=1$ gives

$$(s\cdot u)(f) \;=\; u\Big(-f - x\cdot f'\Big) \;=\; -u(f) - (x\cdot u)(f') \;=\; -u(f) + \Big(\frac{d}{dx}(x\cdot u)\Big)(f)$$

$$= \; -u(f) + (1\cdot u + x\cdot u')(f) \;=\; \Big(x\frac{d}{dx}u\Big)(f)$$

as claimed. ///

[3.3] Theorem: Up to constant multiples, the principal-value distribution η is the unique *odd* distribution of positive-homogeneity degree -1. More generally, except for $s=-n$ with $\varepsilon = (-1)^{n+1}$ with $n=1,2,\ldots$, there is a unique (up to constant multiples) distribution of parity ε and of positive-homogeneity degree s.

Proof: Fix parity $\varepsilon = \pm$, and let V be the set of test functions of parity ε vanishing to infinite order at 0. This is the closure, in the space $\mathcal{D}_\varepsilon = \mathcal{D}_\varepsilon(\mathbb{R})$ of test functions of parity ε, of the space of test functions on $\mathbb{R}$ with support not containing 0 and of parity ε. On $C^\infty_c(0,+\infty)$ there is a unique continuous functional on $C^\infty_c(0,+\infty)$ positive homogeneous of degree s (as distribution), by the appendix, given by integration against $|x|^s$. Given a choice of parity ε, there is a unique extension to a continuous linear functional v^ε_s on $\mathcal{D}^\varepsilon_o$, and of positive-homogeneity degree s. Dualize the short exact sequence

$$0 \longrightarrow V \longrightarrow \mathcal{D}_\varepsilon \longrightarrow \mathcal{D}_\varepsilon/V \longrightarrow 0$$

to a short exact sequence (invoking Hahn-Banach for exactness at the right-most joint)

$$0 \longrightarrow (\mathcal{D}_\varepsilon/V)^* \longrightarrow \mathcal{D}^*_\varepsilon \longrightarrow V^* \longrightarrow 0$$

The quotient $\mathcal{D}_\varepsilon/V$ is identifiable with germs of smooth functions of parity ε supported at 0 modulo germs of smooth functions vanishing identically at 0 of parity ε. The dual is the collection of distributions supported at 0 of parity ε, which by the theory of Taylor-Maclaurin series (see appendix) consists of finite linear combinations of Dirac δ and its derivatives, or parity ε. Of course, δ is even, δ' is odd, and so on.

Let $T = x\frac{d}{dx} - s$, and consider the very small (vertical) complex of short exact sequences

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\mathcal{D}_\varepsilon/V)^* & \longrightarrow & \mathcal{D}_\varepsilon^* & \longrightarrow & V^* & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow T & & \downarrow T & & \\ 0 & \longrightarrow & (\mathcal{D}_\varepsilon/V)^* & \longrightarrow & \mathcal{D}_\varepsilon^* & \longrightarrow & V^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

The corresponding long (co-) homology sequence is

$$0 \to \ker T\Big|_{(\mathcal{D}_\varepsilon/V)^*} \to \ker T\Big|_{\mathcal{D}_\varepsilon^*} \to \ker T\Big|_{V^*} \to \frac{(\mathcal{D}_\varepsilon/V)^*}{T((\mathcal{D}_\varepsilon/V)^*)} \to \frac{\mathcal{D}_\varepsilon^*}{T(\mathcal{D}_\varepsilon^*)} \to \frac{V^*}{T(V^*)} \to 0$$

The functional v_s^ε is in $\ker T|_{V^*}$, and we hope to extend it uniquely to a functional in $\ker T|_{\mathcal{D}_\varepsilon^*}$. It would suffice that $\ker T_{(\mathcal{D}_\varepsilon/V)^*} = 0$ and $T((\mathcal{D}_\varepsilon/V)^*) = (\mathcal{D}_\varepsilon/V)^*$, that is, T is an isomorphism on distributions supported on $\{0\}$ of parity ε and positive-homogeneity degree s. The only failures are $s = -n$ with $\varepsilon = (-1)^{n+1}$, because

$$\Big(x\frac{d}{dx} - (-n)\Big)\delta^{(n)} \;=\; 0 \qquad\qquad (\text{for } n = 1, 2, \ldots)$$

For the principal-value functional, although $s = -1$ does occur as a pole for parity $+1$, the parity is $\varepsilon = -1$, so there is no obstruction to the extension, and it is unique. ///

For $\mathrm{Re}(s) > -1$, let u_s^ε be the distributions given by integration against $|x|^s$ for $\varepsilon = +1$ and $\mathrm{sgn}\, x \cdot |x|^s$ for $\varepsilon = -1$. For $\mathrm{Re}(s) > -1$, these are locally integrable, and of respective parities $\varepsilon = \pm 1$ and of positive-homogeneity degree s.

[3.4] Corollary: The distributions u_s^ε have meromorphic continuations to tempered-distribution-valued functions of $s \in \mathbb{C}$, with only simple poles. For $\varepsilon = +1$, the poles are at $-n = -1, -3, -5, \ldots$, with residues constant multiples of $\delta^{(n)}$. For $\varepsilon = -1$, the poles are at $-n = -2, -4, -6, \ldots$, with residues constant multiples of $\delta^{(n)}$. The meromorphic continuations maintain the positive-homogeneity and parity. For $s \neq -1$, $u_s^\varepsilon = \frac{1}{s+1}\frac{d}{dx}u_{s+1}^{-\varepsilon}$. For $s = -1$,

$$\frac{\partial}{\partial x} u_0^{-1} \;=\; 2 \cdot \delta \qquad \text{and} \qquad \frac{\partial}{\partial x}\Big(\frac{\partial}{\partial s} u_s^{+1}\Big|_{s=0}\Big) \;=\; u_{-1}^{-1} \;=\; \eta$$

Proof: The functionals v_s^ε on V are *entire*, since there is no convergence issue for test functions vanishing to infinite order at 0. Thus, at pole $-n$, $\lim_{s\to -n}(s+n)u_s^\varepsilon$ vanishes on test functions vanishing to infinite order at 0. Thus, the residue $\lim_{s\to -n}(s+n)u_s^\varepsilon$ is a distribution supported on $\{0\}$. It has the same parity ε, and is positive-homogeneous of degree $-n$, in the distributional sense. Thus, there is *no* pole unless $\varepsilon = (-1)^n$, for $n = 0, -1, -2, \ldots$.

For $\mathrm{Re}(s) \gg 1$, the integrate-against functionals u_s^ε given by $|x|^s$ for $\varepsilon = +1$ and $\mathrm{sgn} x \cdot |x|^s$ for $\varepsilon = -1$ are of (distributional) positive-homogeneity degree s, and parity ε. For each $\mathrm{Re}(s) \gg 1$, u_s^ε is the unique extensions of the distributions v_s^ε, up to constants possibly depending on s. By the vector-valued form of the identity principle from complex analysis, the same homogeneity and parity properties hold for any meromorphic continuation. From $x\frac{d}{dx}u_s^\varepsilon = s \cdot u_s^\varepsilon$,

$$\frac{d}{dx}u_s^{-\varepsilon} \;=\; \frac{1}{x}\cdot s \cdot u_s^{-\varepsilon} \;=\; s \cdot u_{s-1}^\varepsilon$$

Thus, $u_s^\varepsilon = \frac{1}{s+1}\frac{d}{dx}u_{s+1}^{-\varepsilon}$. Since $u_{s+1}^{-\varepsilon}$ is a holomorphic distribution-valued function in $\mathrm{Re}(s) > -2$, this extends u_s^ε to $\mathrm{Re}(s) > -2$, except for possible pole at $s = -1$. By induction, $u_s^\pm$ has a meromorphic continuation to all of $\mathbb{C}$ except for possible (simple) poles at negative integers. As above, the only possible residues are distributions supported at 0, which can only occur at non-positive even integers, or non-positive odd integers, depending on parity.

For $\varepsilon = +1$, the relation $\frac{d}{dx}u_s^{-\varepsilon} = s \cdot u_{s-1}^\varepsilon$ evaluated at $s = 0$ yields the *residue* of u_s^{+1} at -1, namely, a constant multiple of δ. For $\varepsilon = -1$, there is no pole, and evaluation of the relation at $s = 0$ just gives $\frac{d}{dx}1 = 0$. Thus, we differentiate in s before evaluation: the relation $\frac{\partial}{\partial x}u_s^{-\varepsilon} = s \cdot u_{s-1}^\varepsilon$ gives

$$\frac{\partial}{\partial x}\Big(\frac{\partial}{\partial s}\Big|_{s=0} u_s^{+1}\Big) \;=\; \frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial x}u_s^{+1} \;=\; \frac{\partial}{\partial s}\Big|_{s=0}\Big(s \cdot u_{s-1}^{-1}\Big) \;=\; (u_{s-1}^{-1} + s \cdot \frac{\partial}{\partial s}u_s^{-1})\Big|_{s=0} \;=\; u_{-1}^{-1}$$

Of course,

$$\frac{\partial}{\partial s}\Big|_{s=0} u_s^{+1} \;=\; \frac{\partial}{\partial s}\Big|_{s=0}|x|^s \;=\; (|x|^s \cdot \log|x|)\Big|_{s=0} \;=\; \log|x|$$

This completes the discussion. ///

[3.5] Remark: In particular, the meromorphic continuations are *tempered* distributions. That is, all positive-homogeneous distributions are tempered. Also, away from 0, homogeneous distributions are given locally by smooth functions.

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4. *Hilbert transforms*

The most immediate description of the Hilbert transform $\mathcal{H}f$ is

$$(\mathcal{H}f)(y) \;=\; \frac{1}{\pi}P.V.\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy \;=\; \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{f(y)}{x-y}\,dy$$

From the previous discussion of the principal-value functional η, $\mathcal{H}f$ exists at least as a point-wise function. In fact, with $(L_y)f(x) = f(x-y)$, $y \to L_y f$ is a smooth $\mathscr{S}$-valued function, so $f \to \eta(L_y f)$ is in $C^\infty(\mathbb{R})$, but growth properties are unclear.

The usual heuristic is

$$P.V.\int_{\mathbb{R}}\frac{f(y)}{x-y}\,dy \;=\; (\eta * f)(x)$$

so

$$\mathcal{H}f \;=\; \frac{1}{\pi}\eta * f \qquad\qquad (\text{for } f \in \mathscr{S}(\mathbb{R}))$$

Granting this for a moment, taking Fourier transform would seem to give

$$(\mathcal{H}f)\widehat{} \;=\; \frac{1}{\pi}\widehat{\eta}\cdot\widehat{f}$$

The principal-value functional η is a tempered distribution, so its Fourier transform makes sense at least as a tempered distribution.

[4.1] Claim: The Fourier transform of an odd/even distribution of positive-homogeneity degree s is odd/even of positive-homogeneity degree $-(s+1)$.

Proof: Since Schwartz functions are dense, the interaction of dilation and Fourier transform can be examined via functions with point-wise values and Fourier transforms defined by literal integrals, although none of these can be homogeneous. It is immediate by changing variables that parity is preserved. With $(f\circ t)(x) = f(tx)$, for *functions* f,

$$(f\circ t)\widehat{}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}f(tx)\,dx \;=\; \frac{1}{t}\int_{\mathbb{R}} e^{-2\pi i\xi x/t}f(tx)\,dx \;=\; \frac{1}{t}(\widehat{f}\circ\frac{1}{t})(\xi)$$

That is,

$$\widehat{f} \circ t \;=\; \frac{1}{t} \cdot f \circ \frac{1}{t}$$

Thus, without trying to use pointwise sense of a tempered distribution,

$$(\widehat{u} \circ t)(f) \;=\; \widehat{u}(\frac{1}{t}\, f \circ \frac{1}{t}) \;=\; \frac{1}{t} u\Big((f \circ \frac{1}{t})\widehat{\;}\Big) \;=\; \frac{1}{t} u\Big(t \cdot \widehat{f} \circ t\Big) \;=\; \frac{1}{t}(u \circ \frac{1}{t})(\widehat{f})$$

$$=\; \frac{1}{t}\Big((\frac{1}{t})^s \cdot u\Big)(\widehat{f}) \;=\; t^{-(s+1)} \cdot u(\widehat{f}) \;=\; t^{-(s+1)} \cdot \widehat{u}(f)$$

as claimed. ///

[4.2] Corollary: The principal-value distribution η, has Fourier transform has degree 0 and of odd parity. In particular, $\widehat{\eta} = -i\pi \operatorname{sgn} x$.

Proof: By uniqueness of distributions of a given degree and parity, it suffices to evaluate $\widehat{\eta}$ and sgn$x$ on a given odd Schwartz function, for example, $x \to xe^{-\pi x^2}$. On one hand,

$$\widehat{\eta}(xe^{-\pi x^2}) \;=\; \eta\Big((xe^{-\pi x^2})\widehat{\;}\Big) \;=\; \eta\Big(i^{-1} xe^{-\pi x^2}\Big) \;=\; i^{-1} \int_{\mathbb{R}} \frac{xe^{-\pi x^2}}{x}\, dx \;=\; i^{-1} \int_{\mathbb{R}} e^{-\pi x^2}\, dx \;=\; i^{-1}$$

On the other,

$$\int_{\mathbb{R}} \operatorname{sgn} x \cdot xe^{-\pi x^2}\, dx \;=\; \int_{\mathbb{R}} |x|\, e^{-\pi x^2}\, dx \;=\; 2\int_0^\infty x\, e^{-\pi x^2}\, dx \;=\; 2\int_0^\infty x^2\, e^{-\pi x^2}\, \frac{dx}{x} \;=\; \int_0^\infty x\, e^{-\pi x}\, \frac{dx}{x}$$

$$=\; \pi^{-1} \int_0^\infty x\, e^{-x}\, \frac{dx}{x} \;=\; \pi^{-1}\Gamma(1) \;=\; \pi^{-1}$$

Thus, $\widehat{\eta} = -i\pi \operatorname{sgn} x$. ///

With the factors of π cancelling, this suggests that we could prove

[4.3] Corollary: $\mathcal{H} f \;=\; \Big(-i \operatorname{sgn} x \cdot \widehat{f}\Big)^\vee$ *[... iou ...]*

[4.4] Corollary: The Hilbert transform extends by continuity to an isometric isomorphism of L^2 to itself.

Proof: Assuming the corollary, since Fourier tranform is an isometry of L^2 to itself, and multiplication by *sgn* is an isometry, the Hilbert transform is an L^2-isometry on $\mathscr{S} \cap L^2$, and extends by continuity to L^2. ///

[4.5] Corollary: $\mathcal{H} \circ \mathcal{H} f(x) \;=\; f(-x)$ for $f \in \mathscr{S}$ or $f \in L^2$. ///

But what meaning should convolution of $f \in \mathscr{S}$ with $\eta \in \mathscr{S}^*$ have? [iou]

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5. *Examples*

[5.1] Example: $\mathcal{H} \sin x = \Big(-i \operatorname{sgn} x \cdot \widehat{\sin}\Big)^\vee = \Big(-i \operatorname{sgn} x \cdot \dfrac{\delta_1 - \delta_{-1}}{2i}\Big)^\vee = \Big(-\dfrac{\delta_1 + \delta_{-1}}{2}\Big)^\vee = -\cos x$

[5.2] Example: Since $\widehat{\operatorname{ch}_{[-1,1]}} = \frac{\sin 2\pi\xi}{\pi\xi}$,

$$\mathcal{H}\frac{\sin 2\pi x}{\pi x} \;=\; \Big(-i \operatorname{sgn} x \cdot \operatorname{ch}_{[-1,1]}\Big)^\vee \;=\; i\Big(\operatorname{ch}_{[-1,0]} - \operatorname{ch}_{[0,1]})\Big)^\vee \;=\; i\frac{e^{-2\pi i\xi} - 1 - 1 + e^{2\pi i\xi}}{2\pi i\xi} \;=\; \frac{\cos 2\pi\xi - 1}{\pi\xi}$$

[5.3] Example: $\mathcal{H}|x|^s$ *[... iou ...]*

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6. ...

[... iou ...]

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7. *Appendix: uniqueness of equivariant functionals*

[7.1] Theorem: Let G be a topological group. Let $V \subset C_c^o(G)$ be a quasi-complete locally convex topological vector space of complex-valued functions on G stable under left and right translations, and containing an approximate identity $\{\varphi_i\}$. Then there is a unique *right* G-invariant continuous functional on V (up to constant multiples): it is

$$f \to \int_G f(g)\,dg \qquad \text{(with right translation-invariant measure)}$$

Proof: Let $T_g f(y) = f(yg)$ be right translation. With right translation-invariant measure dg on G, since $\int_G \varphi_i(g)\,dg = 1$ and φ_i is non-negative, $\varphi_i(g)\,dg$ is a probability measure on the (compact) support of φ_i. Thus, for any $f \in V$, we have a V-valued Gelfand-Pettis integral

$$T_{\varphi_i} f \;=\; \int_G \varphi_i(g)\,T_g f\;dg \;\in\; \text{closure of convex hull of } \{\varphi_i(g) f : g \in G\} \;\subset\; V$$

The assumption $V \subset C_c^o(G)$ is meant to entail that the translation action of G on V is *continuous*, meaning that $G \times V \to V$ by $g \times f \to T_g f$ is (jointly) continuous. By continuity, given a neighborhood N of 0 in V, we have $T_{\varphi_i} f \in f + N$ for all sufficiently large i. That is, $T_{\varphi_i} f \to f$. For a right-invariant (continuous) functional u on V,

$$u(f) \;=\; \lim_i u\left(g \to \int_G \varphi_i(h)\,f(gh)\,dh\right)$$

This is

$$u\left(g \to \int_G f(hg)\,\varphi_i(h^{-1})\,dh\right) \;=\; u\left(g \to \int_G f(h)\,\varphi_i(gh^{-1})\,dh\right)$$

by replacing h by hg^{-1}. By properties of Gelfand-Pettis integrals, and since f is guaranteed to be a compactly-supported continuous function, we can move the functional u inside the integral: the above becomes

$$\int_G f(h)\,u\left(g \to \varphi_i(gh^{-1})\right)\,dh$$

Using the *right* G-invariance of u the evaluation of u with right translation by h^{-1} gives

$$\int_G f(h)\,u(g \to \varphi_i(g))\,dh \;=\; u(\varphi_i) \cdot \int_G f(h)\,dh$$

By assumption the latter expressions approach $u(f)$ as $i \to \infty$. For f so that the latter integral is non-zero, we see that the limit of the $u(\varphi_i)$ exists, and then we conclude that $u(f)$ is a constant multiple of the indicated integral with right Haar measure. ///

[7.2] Corollary: In the situation of the theorem, for a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$, suppose that V is stable under multiplication by the function χ. Then there is a unique continuous functional λ on V such that $\lambda(T_g f) = \chi(g) \cdot \lambda(f)$ for all $g \in G$.

Proof: Let u be the G-invariant functional of the theorem, and put $\lambda(f) = u(\chi \cdot f)$. ///

8. *Appendix: distributions supported at* $\{0\}$

[8.1] **Theorem:** A distribution u with *support* $\{0\}$ is a (finite) linear combination of Dirac's δ and its derivatives.

Recall: the *support* of a distribution u is the *complement* of the *union* of all open sets $U \in \mathbb{R}^n$ such that

$$u(f) \;=\; 0 \qquad\qquad (\text{for } f \in \mathcal{D}_K \text{ with compact } K \subset U)$$

Proof: Since the space $\mathcal{D}$ of test functions on $\mathbb{R}^n$ is $\mathcal{D} = \mathrm{colim}_K \mathcal{D}_K$, it suffices to classify u in $\mathcal{D}_K^*$ with support $\{0\}$.

A continuous linear map T from a *limit* of Banach spaces (such as $\mathcal{D}_K$) to a normed space (such as $\mathbb{C}$) factors through a limitand, under the mild assumption that the images of the limit in the limitands are *dense*. Thus, there is an *order* $k \geq 0$ such that u factors through

$$C_K^k \;=\; \{f \in C^k(K) :\; f^{(\alpha)} \text{ vanishes on } \partial K \text{ for all } \alpha \text{ with } |\alpha| \leq k\}$$

Fix a smooth compactly-supported function ψ identically 1 on a neighborhood of 0, bounded between 0 and 1, and (necessarily) identically 0 outside some (larger) neighborhood of 0. For $\varepsilon > 0$ let

$$\psi_\varepsilon(x) \;=\; \psi(\varepsilon^{-1} x)$$

Since the support of u is just $\{0\}$, for all $\varepsilon > 0$ and for all $f \in \mathcal{D}(\mathbb{R}^n)$ the support of $f - \psi_\varepsilon \cdot f$ does not include 0, so

$$u(\psi_\varepsilon \cdot f) \;=\; u(f)$$

Thus, for some constant C (depending on k and K, but not on f)

$$|\psi_\varepsilon f|_k \;=\; \sup_{x \in K} \sup_{|\alpha| \leq k} |(\psi_\varepsilon f)^{(\alpha)}(x)| \;\leq\; C \cdot \sup_{|i| \leq k} \sup_{x} \sup_{0 \leq j \leq i} \varepsilon^{-|j|} \left| \psi^{(j)}(\varepsilon^{-1} x)\, f^{(i-j)}(x) \right|$$

For f vanishing to order k at 0, that is, $f^{(\alpha)}(0) = 0$ for all multi-indices α with $|\alpha| \leq k$, on a fixed neighborhood of 0, by a Taylor-Maclaurin expansion, for some constant C

$$|f(x)| \;\leq\; C \cdot |x|^{k+1}$$

and, generally, for α^{th} derivatives with $|\alpha| \leq k$,

$$|f^{(\alpha)}(x)| \;\leq\; C \cdot |x|^{k+1-|\alpha|}$$

For some constant C

$$|\psi_\varepsilon f|_k \;\leq\; C \cdot \sup_{|i| \leq k} \sup_{0 \leq j \leq i} \varepsilon^{-|j|} \cdot \varepsilon^{k+1-|i|+|j|} \;\leq\; C \cdot \varepsilon^{k+1-|i|} \;\leq\; C \cdot \varepsilon^{k+1-k} \;=\; C \cdot \varepsilon$$

Thus, for all $\varepsilon > 0$, for smooth f vanishing to order k at 0,

$$|u(f)| \;=\; |u(\psi_\varepsilon f)| \;\leq\; C \cdot \varepsilon$$

Thus, $u(f) = 0$ for such f.

That is, u is 0 on the intersection of the kernels of δ and its derivatives $\delta^{(\alpha)}$ for $|\alpha| \leq k$. Generally,

[8.2] Proposition: A continuous linear function $\lambda \in V^*$ vanishing on the intersection of the kernels of a finite collection $\lambda_1, \ldots, \lambda_n$ of continuous linear functionals on V is a linear combination of the λ_i.

Proof: The linear map

$$q : V \longrightarrow \mathbb{C}^n \quad \text{by} \quad v \longrightarrow (\lambda_1 v, \ldots, \lambda_n v)$$

is *continuous* since each λ_i is continuous, and λ factors through q, as $\lambda = L \circ q$ for some linear functional L on $\mathbb{C}^n$. We know all the linear functionals on $\mathbb{C}^n$, namely, L is of the form

$$L(z_1, \ldots, z_n) \;=\; c_1 z_1 + \ldots + c_n z_n \qquad\qquad \text{(for some constants } c_i\text{)}$$

Thus,

$$\lambda(v) \;=\; (L \circ q)(v) \;=\; L(\lambda_1 v, \ldots, \lambda_n v) \;=\; c_1 \lambda_1(v) + \ldots + c_n \lambda_n(v)$$

expressing λ as a linear combination of the λ_i. ///

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9. *Appendix: the snake lemma*

The *snake lemma* produces a long exact sequence from a short exact sequence of complexes, where in this context a *complex* is a chain of homomorphisms

$$\ldots \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} M_{-1} \xrightarrow{f_{-1}} \ldots$$

where $\operatorname{Im} f_i \subset \ker f_{i_1}$ for all indices. [1] For a slightly specialized version [2] of the snake lemma, consider a short exact sequences of very small complexes of the form $0 \to A \to A \to 0$, $0 \to B \to B \to 0$, and $0 \to C \to C \to 0$, with the complexes themselves now written vertically:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow T & & \downarrow T & & \downarrow T & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We assume that the maps $T : A \to A$, $T : B \to B$, and $T : C \to C$ make the squares in the latter diagram commute. In the following, one should consider the improbability of defining some sort of map from any part of C to any part of A, given short exact $0 \to A \to B \to C \to 0$:

[9.1] Claim: There is a long exact sequence

$$0 \longrightarrow \ker T|_A \longrightarrow \ker T|_B \longrightarrow \ker T|_C \longrightarrow \frac{A}{TA} \longrightarrow \frac{B}{TB} \longrightarrow \frac{C}{TC} \longrightarrow 0$$

---

[1] There is also an assertion of *naturality*, which we ignore here. See [Weibel 1994] as general reference for these and related ideas.

[2] I first saw this small but important example in [Casselman 1993].

Proof: The most serious issue, which is all we address, is defining the *connecting homomorphism* δ : $\ker T|_C \to A/TA$. For example, the exactness of the left half and of the right half of the long sequence are relatively elementary. [3] Given $c \in \ker T|_C$, we want to determine $a \in A$ well-defined module TA, to eventually fit into the long exact sequence. Let $b \to c$ for some $b \in B$. Since $Tc = 0 \in C$ and the squares commute, $Tb \to 0$ under the map $B \to C$. Thus, by exactness of $0 \to A \to B \to C \to 0$, there is $a \in A$ mapping to Tb. Declare $a = \delta c$. To make δ well-defined, note that the precise ambiguity in choice of $a \in A$ is by TA. ///

Bibliography

[Bedrosian 1962] E. Bedrosian, *A product theorem for Hilbert transforms*, Memoranudm RM-3439-PR, December 1962, Rand Corp., Santa Monica, CA.

[Calderon-Zygmund 1952] A.P. Calderon, A. Zygmund, *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85-139.

[Casselman 1993] W. Casselman, *Extended automorphic forms on the upper half-plane*, Math. Ann. **296** (1993), 755-762.

[Chaudhury 2012] K. N. Chaudhury, *L^p-boundedness of the Hilbert transform*, arXiv:0999.1426v9 [cs.IT] 2 Oct 2012.

[Fefferman 1971] C. Fefferman, *Characterization of bounded mean oscillation*, Bull. AMS **77** (1971), 587-588.

[Fefferman-Stein 1972] C. Fefferman, E. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137-193.

[Garrett 2016a] P. Garrett, *Vector-valued integrals*,
http://www.math.umn.edu/~garrett/m/fun/notes-2016-17/vector-valued-integrals.pdf

[Garrett 2016b] P. Garrett, *Holomorphic vector-valued functions*,
http://www.math.umn.edu/~garrett/m/fun/notes-2016-17/holomorphic-vector-valued.pdf

[Gelfand-Silov 1964] I.M. Gelfand, G.E. Silov, *Generalized Functions, I: Properties and Operators*, Academic Press, NY, 1964.

[Pichorides 1972] S. K. Pichorides, *On the best values of the constants in the theorems of M. Riesz, Zygmund, and Kolmogorov*, Studia Math. **44** (1972), 165-179.

[Plemelj 1908] J. Plemelj, *Ein Egänzungssatz zur Cauchyschen Integraldarstellung analytischer Funktionen, Randwerte betreffend*, Monatsch. Math. Phys. **19** (1908), 205-10.

[Plemelj 1908b] J. Plemelj, *Riemannsche Funktionenscharen mit gegebener Monodromiegruppe*, Monatsch. Math. Phys. **19** (1908), 211-245.

[Riesz 1928] M. Riesz, *Sur les fonctions conjugées*, Math. Z. (1928), 218-244.

[Sokhotskii 1873] Y.W. Sokhotskii, *On definite integrals and functions used in series expansion*, (Russian?) St. Petersburg, 1873.

[Wiki 2015] https://en.wikipedia.org/wiki/Sokhotski-Plemelj_theorem, accessed 08 Oct 2015.

[Stein 1970] E. Stein, *Singular integrals and Differentiability Properties of Functions*, Princeton U. Press,

[3] The exactness of those halves also follows from the slightly-less elementary fact that right adjoints are left exact, and left adjoints are right exact.

1970.

[Weibel 1994] C. Weibel, *An introduction to homological algebra*, Cambridge Univ. Press, 1994.

[Wiki 2014] Wikipedia, *Hilbert transform*, https://en.wikipedia.org/wiki/Hilbert_transform, 20 Dec 2014.

[Woodward-Davies 1952] P.M. Woodward, I.L. Davies, *Information theory and inverse probability in telecommunication*, Proc. Inst. Elec. Engrs. Part III **99** (1952), 37-44.

[Zygmund 1968] A. Zygmund, *Trigonometric Series I, II*, Cambridge, 1968.