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Holomorphic vector-valued functions

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1. *Weak-to-strong differentiability*

A V-valued function $f : [a, c] \to V$ on an interval $[a, c] \subset \mathbb{R}$ is *differentiable* if for every $x_o \in [a, c]$

$$f'(x_o) \;=\; \lim_{x \to x_o} (x - x_o)^{-1} \Big(f(x) - f(x_o) \Big)$$

exists. The function f is *continuously differentiable* when it is differentiable and f' is continuous. A k-times continuously differentiable function is C^k, and a continuous function is C^o.

A V-valued function f is *weakly* C^k when for every $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ is C^k. This sense of *weak differentiability* of a function f does *not* refer to distributional derivatives, but to differentiability of every scalar-valued function $\lambda \circ f$ where $\lambda \in V^*$ for V-valued f.

[1.1] Theorem: For quasi-complete, locally convex V, a *weakly* C^k V-valued function f on an interval $[a, c]$ is *strongly* C^{k-1}.

Proof: To have f be (strongly) differentiable at fixed $b \in [a, c]$ is to have (strong) *continuity* at b of

$$g(x) \;=\; \frac{f(x) - f(b)}{x - b} \qquad\qquad (\text{for } x \neq b)$$

Weak C^2-ness of f implies that every $\lambda \circ g$ extends to a C^1 scalar-valued function on $[a, c]$. We need to get from this to a (strongly) continuous extension of g to the whole interval.

The (strong) continuity of f' will follow from consideration of the function of two variables (initially for $x \neq y$)

$$g(x, y) \;=\; \frac{f(x) - f(y)}{x - y}$$

The weak C^2-ness of f assures that g extends to a weakly C^1 function on $[a, c] \times [a, c]$. In particular, the function $x \to g(x, x)$ of (the extended) g is weakly C^1, and $x \to g(x, x)$ is $f'(x)$, so f' is weakly C^1. From above, f' is (strongly) C^o. Suppose that we already know that f is C^ℓ, for $\ell < k - 1$. As the ℓ^{th} derivative $g = f^{(\ell)}$ of f is weakly C^2, it is (strongly) C^1 by the first part of the argument. That is, f is $C^{\ell+1}$. ///

2. *Holomorphic vector-valued functions*

Let V be a quasi-complete, locally convex topological vector space. A V-valued function f on a non-empty open set $\Omega \subset \mathbb{C}$ is (strongly) *complex-differentiable* when $\lim_{z\to z_o} \big(f(z) - f(z_o)\big)/(z - z_o)$ exists (in V) for all $z_o \in \Omega$, where $z \to z_o$ specificially means for *complex* z approaching z_o. The function f is *weakly holomorphic* when the $\mathbb{C}$-valued functions $\lambda \circ f$ are holomorphic for all λ in V^*. The useful version of vector-valued *meromorphy* of f at z_o is that $(z - z_o)^n \cdot f(z)$ extends to a vector-valued *holomorphic* function at z_o for some n. After some preparation, we will prove

[2.1] Theorem: *Weakly* holomorphic V-valued functions f are *continuous. (Proof below.)* ///

[2.2] Corollary: Weakly holomorphic V-valued functions are (strongly) holomorphic. The Cauchy integral formula applies:

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \qquad\qquad \text{(as Gelfand-Pettis } V\text{-valued integral)}$$

Proof: Since $f(z)$ is continuous, the integral

$$I(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

exists as a Gelfand-Pettis integral. Thus, for any $\lambda \in V^*$

$$\lambda\big(I(z)\big) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{(\lambda\circ f)(w)}{w-z}\,dw \;=\; (\lambda\circ f)(z)$$

by the holomorphy of $\lambda \circ f$. By Hahn-Banach, linear functionals separate points, so $I(z) = f(z)$, giving the Cauchy integral formula for f itself.

To prove (strong) complex-differentiability of f at z_o, take $z_o = 0$ and use $f(0) = 0$, for convenience. There is a disk $|z| < 3r$ such that for every $\lambda \in V^*$

$$F_\lambda(z) \;=\; \frac{(\lambda\circ f)(z)}{z} \qquad\qquad \text{(on } 0 < |z| < r\text{)}$$

extends to a holomorphic function on $|z| < r$. Continuity of f assures existence of

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w}\,\frac{dw}{w-z}$$

By Cauchy theory for $\mathbb{C}$-valued functions, and Gelfand-Pettis,

$$\lambda\Big(\frac{f(z)}{z}\Big) \;=\; F_\lambda(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{(\lambda\circ f)(w)}{w}\,\frac{dw}{w-z} \;=\; \lambda\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w}\,\frac{dw}{w-z}\Big)$$

Since functionals separate points,

$$\frac{f(z)}{z} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w}\,\frac{dw}{w-z}$$

From

$$\frac{1}{w(w-z)} \;=\; \frac{1}{w^2} + \frac{z}{w^2(w-z)}$$

we have

$$\frac{f(z)}{z} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^2}\,dw + z\cdot\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^2(w-z)}\,dw$$

Using the continuity of f, given a convex balanced neighborhood U of 0 in V, the compact set $K = \{f(w) : |w| = 2r\}$ is contained in some multiple t_oU of U. Thus, for $|z| < r$,

$$\frac{f(z)}{z} - \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^2}\,dw \quad\in\quad |z|\cdot\frac{1}{(2r)^2\,r}\cdot t_oU$$

so $\lim_{z\to 0} f(z)/z$ exists. Since $f(0) = 0$,

$$\lim_{z\to z_o}\frac{f(z)-f(z_o)}{z-z_o} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{(w-z_o)^2}$$

giving the complex differentiability of f. ///

[2.3] Corollary: The usual Cauchy-theory integral formulas apply. In particular, weakly holomorphic f is (strongly) infinitely differentiable, in fact expressible as a convergent power series with coefficients given by Cauchy's formulas:

$$f(z) \;=\; \sum_{n\ge 0} c_n\,(z-z_o)^n \qquad\text{with}\qquad c_n \;=\; \frac{f^{(n)}(z_o)}{n!} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_o)^{n+1}}\,dw$$

for γ a path with winding number $+1$ around z_o.

Proof: Without loss of generality, treat $z_o = 0$, and $|z| < \rho|w|$ with $\rho < 1$, and $|w| = r$. The expansion

$$\frac{1}{w-z} \;=\; \frac{1}{w}\,\frac{1}{1-\frac{z}{w}} \;=\; \frac{1}{w}\Big(1+\frac{z}{w}+\big(\frac{z}{w}\big)^2+\ldots+\big(\frac{z}{w}\big)^N+\frac{(z/w)^{N+1}}{1-\frac{z}{w}}\Big)$$

combined with an integration around γ against $f(w)$, and the basic Cauchy integral formula, give

$$f(z) \;=\; \sum_{n=0}^{N}\frac{1}{2\pi i}\int_\gamma \frac{f(w)\,dw}{w^{n+1}}\cdot z^n \;+\; \frac{1}{2\pi i}\int_\gamma \frac{1}{w^{N+1}}\,\frac{f(w)\,dw}{w-z}\cdot z^{N+1}$$

Much as in the previous proof, given a convex balanced neighborhood U of 0 in V, the compact set $K = \{f(w) : |w| = r\}$ is contained in some multiple t_oU of U, and

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{w^{N+1}}\,\frac{f(w)\,dw}{w-z}\cdot z^{N+1} \quad\in\quad \frac{1}{r^{N+1}}\cdot t_oU\cdot\frac{1}{r(1-\rho)}\cdot(\rho r)^{N+1} \;=\; U\frac{t_o}{r(1-\rho)}\rho^{N+1}$$

Since $0<\rho<1$, $\rho^{N+1}/r(1-\rho) < 1$ for sufficiently large N, so the leftover term is inside given U. ///

An appendix discusses the differentiability of power series with coefficients in topological vector spaces.

The next section collects some important corollaries of the main result, prior to preparation for the proof that weak holomorphy implies *continuity*,

3. *Holomorphic Hol*(Ω, V)*-valued functions*

The vector-valued versions of Cauchy's formulas have useful corollaries. First, recall some aspects of the classical scalar-valued case.

For open $\phi \neq \Omega \subset \mathbb{C}$, give the space $\mathrm{Hol}(\Omega)$ of holomorphic functions on Ω the topology given by the seminorms $\mu_K(f) = \sup_{z\in K} |f(z)|$ for compacts $K \subset \Omega$.

[3.1] Claim: $\mathrm{Hol}(\Omega)$ is a Fréchet space.

Proof: Let $\{f_n\}$ be a Cauchy sequence in that topology. As in [13.5], the pointwise limit $f(z) = \lim_n f_n(z)$ is at least *continuous*. Then, for a small circle γ inside Ω and enclosing z,

$$f(z) \;=\; \lim_n f_n(z) \;=\; \lim_n \frac{1}{2\pi i}\int_\gamma \frac{f_n(w)}{w-z}\,dw$$

Since γ is compact and the limit is uniformly approached on compacts, this gives

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \lim_n \frac{f_n(w)}{w-z}\,dw \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

Direct estimates (simpler than in the previous section) show that the latter integral is complex-differentiable in w. ///

Let V be quasi-complete, locally convex, with topology given by seminorms $\{\nu\}$. The space $\mathrm{Hol}(\Omega, V)$ of holomorphic V-valued functions on Ω has the natural topology given by seminorms

$$\mu_{\nu,K}(f) \;=\; \sup_{z\in K} \nu(f(z)) \qquad \text{(compacts } K\subset\Omega\text{, seminorms } \nu \text{ on } V)$$

This topology is obviously the analogue of the sups-on-compacts seminorms on scalar-valued holomorphic functions, and there is the analogous corollary of the vector-valued Cauchy formulas:

[3.2] Corollary: $\mathrm{Hol}(\Omega, V)$ is locally convex, quasi-complete. ///

Proof: Let $\{f_n\}$ be a bounded Cauchy net. Just as in the scalar case, the pointwise limits $\lim_n f_n(z)$ exist. The same three-epsilon argument as for scalar-valued functions will show that the pointwise limit exists and is continuous, as follows. First, using compact $K = \{z\}$, the value $\mu_{\{z\},\nu}(f)$ is just $\nu(f(z))$. Thus, by quasi-completeness of V, for each fixed z the bounded Cauchy net $f_n(z)$ converges to a value $f(z)$. Given $\varepsilon > 0$ and $z_o \in \Omega$, let K be a compact neighborhood of z_o, and take N sufficiently large so that $\nu(f_m(z) - f_n(z')) < \varepsilon$ for all $z, z' \in K$ and all $m, n \geq N$. Then

$$\mu_{K,\nu}(f(z) - f(z_o)) \;\leq\; \mu_{K,\nu}(f(z) - f_n(z)) + \mu_{K,\nu}(f_n(z) - f_n(z_o)) + \mu_{K,\nu}(f_n(z_o) - f(z_o)) \;\leq\; 3\varepsilon$$

proving the continuity of the pointwise limit. Then, as in the previous scalar-valued argument, the vector-valued Cauchy formula gives, for a small circle γ inside Ω and enclosing z,

$$f(z) \;=\; \lim_n f_n(z) \;=\; \lim_n \frac{1}{2\pi i}\int_\gamma \frac{f_n(w)}{w-z}\,dw$$

with Gelfand-Pettis integrals. Since γ is compact and the limit is uniformly approached on compacts, this gives

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \lim_n \frac{f_n(w)}{w-z}\,dw \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw$$

Again, the differentiability of latter integral is directly verifiable, and f is holomorphic. ///

It is occasionally useful to iterate the previous ideas: A V-valued function $f(z,w)$ on a non-empty open subset $\Omega \subset \mathbb{C}^2$ is *complex analytic* when it is locally expressible as a convergent power series in z and w, with coefficients in V. The two-variable version of the discussion of convergence of power series with coefficients in V in the appendix succeeds without incident in the two-variable case. [1]

[3.3] Corollary: Let $f(z,w)$ be complex-analytic $\mathbb{C}$-valued in two variables, on a domain $\Omega_1 \times \Omega_2 \subset \mathbb{C}^2$. Then the function $w \longrightarrow (z \to f(z,w))$ is a holomorphic $\mathrm{Hol}(\Omega_1)$-valued function on Ω_2.

Proof: The issue is the uniformity in z in compacts K of the limit

$$\lim_{h\to 0} \frac{f(z,w+h)-f(z,w)}{h}$$

Using the scalar-valued Cauchy integral, for a small circle γ about w, letting f_2 be the partial derivative of f with respect to its second argument,

$$\frac{f(z,w+h)-f(z,w)}{h} - f_2(z,w) \;=\; \frac{1}{2\pi i}\int_\gamma f(z,\zeta)\Big(\frac{\frac{1}{\zeta-(w+h)}-\frac{1}{\zeta-w}}{h} - \frac{1}{(\zeta-w)^2}\Big)\,d\zeta$$

$$=\; \frac{1}{2\pi i}\int_\gamma f(z,\zeta)\Big(\frac{1}{(\zeta-(w+h))(\zeta-h)} - \frac{1}{(\zeta-w)^2}\Big)\,d\zeta$$

The two-variable analytic function $z,\zeta \to f(z,\zeta)$ is certainly *continuous* as a function of two variables, so is *uniformly* continuous on compacts $K\times\gamma$. Thus, the limit as $h\to 0$ is approached uniformly. ///

Application of the vector-valued form of Cauchy's integrals gives the same result for $f(z,w)$ taking values in a quasi-complete, locally convex V:

[3.4] Corollary: Let V be quasi-complete, locally convex. Let $f(z,w)$ be complex-analytic V-valued in two variables, on a domain $\Omega_1 \times \Omega_2 \subset \mathbb{C}^2$. Then the function $w \longrightarrow (z \to f(z,w))$ is a holomorphic $\mathrm{Hol}(\Omega_1, V)$-valued function on Ω_2. ///

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4. *Banach-Alaoglu: compactness of polars*

The *polar* U^o of an open neighborhood U of 0 in a topological vector space V is

$$U^o = \{\lambda \in V^* : |\lambda u| \le 1, \;\text{ for all } u \in U\}$$

[4.1] Theorem: *(Banach-Alaoglu)* In the weak dual topology on V^* the polar U^o of an open neighborhood U of 0 in V is *compact*.

Proof: For every v in V there is real t_v sufficiently large such that $v \in t_v \cdot U$, and $|\lambda v| \le t_v$ for $\lambda \in U^o$. Tychonoff gives compactness of the product

$$P \;=\; \prod_{v\in V} \{z \in \mathbb{C} : |z| \le t_v\} \;\subset\; \prod_{v\in V} \mathbb{C}$$

[1] We have no immediate need of subtleties concerning functions of several complex variables, such as Hartogs' theorem that separate analyticity implies joint analyticity.

Map V^* to $\prod_{v\in V} C$ by $j(\lambda) = \{\lambda(v) : v \in V\}$. By design, $j(U^o) \subset P$. To prove the compactness of U^o it suffices to show that the weak dual topology on U^o is identical to the subspace topology on $j(U^o)$ inherited from P, and that $j(U^o)$ is closed in P.

The sub-basis sets

$$\{\lambda \in V^* : |\lambda v - \lambda_o v| < \delta\} \qquad \text{(for } v \in V \text{ and } \delta > 0)$$

for V^* are mapped by j to the sub-basis sets

$$\{p \in P : |p_v - \lambda_o v| < \delta\} \qquad \text{(for } v \in V \text{ and } \delta > 0)$$

for the product topology on P. That is, j maps U^o with the weak star-topology homeomorphically to $j(U^o)$.

To show that $j(U^o)$ is closed in P, consider L in the closure of U^o in P. Given $x, y \in V$, $a, b \in \mathbb{C}$, the sets

$$\{p \in P : |(p-L)_x| < \delta\} \qquad \{p \in P : |(p-L)_y| < \delta\} \qquad \{p \in P : |(p-L)_{ax+by}| < \delta\}$$

are open in P and contain L, so meet $j(U^o)$. Let $\lambda \in j(U^o)$ lie in the intersection of these three sets and $j(U^o)$. Then

$$|aL_x + bL_y - L_{ax+by}| \;\le\; |a|\cdot|L_x - \lambda x| + |b|\cdot|L_y - \lambda y| + |L_{ax+by} - \lambda(ax+by)| + |a\lambda x + b\lambda y - \lambda(ax+by)|$$

$$\le\; |a|\cdot\delta + |b|\cdot\delta + \delta + 0 \qquad \text{(for every } \delta > 0)$$

so L is *linear*. Given $\varepsilon > 0$, for N be a neighborhood of 0 in V such that $x - y \in N$ implies $\lambda x - \lambda y \in N$,

$$|L_x - L_y| \;=\; |L_x - \lambda x| + |L_y - \lambda y| + |\lambda x - \lambda y|\delta + \delta + \varepsilon$$

Thus, L is *continuous*. Also, $|L_x - \lambda x| < \delta$ for all $x \in U$ and all $\delta > 0$, so $L \in j(U^o)$, and $j(U^o)$ is *closed*, giving compactness. ///

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5. *Variant Banach-Steinhaus/uniform boundedness*

This variant of the Banach-Steinhaus (uniform boundedness) theorem is used with Banach-Alaoglu to show that weak boundedness implies boundedness in a locally convex space, the starting point for *weak-to-strong principles*. It uses the version of Baire category for locally compact Hausdorff spaces, rather than complete metric spaces.

[5.1] Theorem: *(Variant Banach-Steinhaus)* Let K be a compact convex set in a topological vectorspace X, and $\mathscr{T}$ a set of continuous linear maps $X \to Y$ from X to another topological vectorspace Y. Suppose that for every *individual* $x \in K$ the collection of images $\mathscr{T}x = \{Tx : T \in \mathscr{T}\}$ is *bounded* in Y. Then $B = \bigcup_{x\in K} \mathscr{T}x$ is bounded in Y.

Proof: Let U, V be balanced neighborhoods of 0 in Y so that $\overline{U} + \overline{U} \subset V$, and let

$$E \;=\; \bigcap_{T\in\mathscr{T}} T^{-1}(\overline{U})$$

By the boundedness of $\mathscr{T}x$, there is a positive integer n such that $\mathscr{T}x \subset nU$, and then $x \in nE$. For every $x \in K$ there is such n, so

$$K \;=\; \bigcup_n (K \cap nE)$$

Since E is closed, the version of the Baire category theorem for locally compact Hausdorff spaces implies that at least one set $K \cap nE$ has non-empty interior in K. For such n, let x_o be an interior point of $K \cap nE$. Pick a balanced neighborhood W of 0 in X such that

$$K \cap (x_o + W) \;\subset\; nE$$

Since K is compact, it is bounded, so $K - x_o$ is bounded, and $K \subset x_o + tW$ for large enough positive real t. Since K is convex, $(1-t^{-1})x + t^{-1}x \in K$ for any $x \in K$ and $t \geq 1$. At the same time,

$$z - x_o \;=\; t^{-1}(x - x_o) \in W \qquad\qquad \text{(for large enough } t\text{)}$$

by the boundedness of K, so $z \in x_o + W$. Thus, $z \in K \cap (x_o + V) \subset nE$. From the definition of E, $TE \subset \overline{U}$, so $T(nE) = nT(E) \subset n\overline{U}$. And $x = tz - (t-1)x_o$ yields

$$Tx \;\in\; tn\overline{U} - (t-1)n\overline{U} \subset tn(\overline{U} + \overline{U})$$

by the balanced-ness of U. Since $\overline{U} + \overline{U} \subset V$, we have $B \subset tnV$. Since V was arbitrary, this proves the boundedness of B. ///

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6. *Weak boundedness implies (strong) boundedness*

[6.1] Theorem: Let V be a locally convex topological vectorspace. A subset E of V is bounded if and only if it is weakly bounded.

Proof: For the proof, we need the notion of *second polar* N^{oo} of an open neighborhood N of 0 in a topological vector space V:

$$N^{oo} \;=\; \{v \in V : |\lambda\, v| \leq 1 \text{ for all } \lambda \in N^o\}$$

where N^o is the polar of N. Conveniently,

[6.2] Claim: *(On second polars)* For V a locally convex topological vectorspace and N a convex, balanced neighborhood of 0, the second polar N^{oo} of N is the closure $\overline{N}$ of N.

Proof: Certainly N is contained in N^{oo}, and in fact $\overline{N}$ is contained in N^{oo} since N^{oo} is closed. By the local convexity of V, Hahn-Banach implies that for $v \in V$ but $v \notin \overline{N}$ there is $\lambda \in V^*$ such that $\lambda v > 1$ and $|\lambda v'| \leq 1$ for all $v' \in \overline{N}$. Thus, λ is in N^o, and every element $v \in N^{oo}$ is in $\overline{N}$, so $N^{oo} = \overline{N}$. ///

Returning to the proof of the theorem: clearly boundedness implies weak boundedness. On the other hand, take E weakly bounded, and U be a neighborhood of 0 in V in the original topology. By local convexity, there is a convex (and balanced) neighborhood N of 0 such that the closure $\overline{N}$ is contained in U.

By the weak boundedness of E, for each $\lambda \in V^*$ there is a bound b_λ such that $|\lambda\, x| \leq b_\lambda$ for $x \in E$. By Banach-Alaoglu the polar N^o of N is compact in V^*. The functions $\lambda \to \lambda x$ are continuous, so by variant Banach-Steinhaus there is a uniform constant $b < \infty$ such that $|\lambda x| \leq b$ for $x \in E$ and $\lambda \in N^o$. Thus, $b^{-1}x$ is in the second polar N^{oo} of N, shown by the previous proposition to be the closure $\overline{N}$ of N. That is, $b^{-1}x \in \overline{N}$. By the balanced-ness of N, $E \subset t\overline{N} \subset tU$ for any $t > b$, so E is bounded. ///

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7. *Proof that weak C^1 implies strong C^o*

The claim below, needed to complete the proof that *weak* C^k implies (strong) C^{k-1}, is an application of the fact that weak boundedness implies boundedness.

[7.1] Claim: Let V be a quasi-complete locally convex topological vector space. Fix real numbers $a \leq b \leq c$. Let g be a V-valued function defined on $[a,b) \cup (b,c]$. Suppose that for $\lambda \in V^*$ the scalar-valued function $\lambda \circ g$ extends to a C^1 function F_λ on the whole interval $[a,c]$. Then $g(b)$ can be chosen such that the extended $g(x)$ is (strongly) continuous on $[a,c]$.

Proof: For each $\lambda \in V^*$, let F_λ be the extension of $\lambda \circ g$ to a C^1 function on $[a, c]$. The differentiability of F_λ implies that for each λ the function

$$\Phi_\lambda(x, y) \;=\; \frac{F_\lambda(x) - F_\lambda(y)}{x - y} \qquad\qquad (\text{for } x \neq y)$$

has a continuous extension $\tilde{\Phi}_\lambda$ to the compact set $[a, c] \times [a, c]$. The image C_λ of $[a, c] \times [a, c]$ under this continuous map is compact in $\mathbb{R}$, so bounded. Thus, the subset

$$\Big\{\frac{\lambda f(x) - \lambda f(y)}{x - y} : x \neq y\Big\} \;\subset\; C_\lambda$$

is *bounded* in $\mathbb{R}$. That is,

$$E \;=\; \Big\{\frac{g(x) - g(y)}{x - y} : x \neq y\Big\} \;\subset\; V$$

is *weakly* bounded. Because weakly bounded implies (strongly) bounded, E is (strongly) bounded. That is, for a balanced, convex neighborhood N of 0 in V, there is t_o such that $(g(x) - g(y))/(x - y) \in tN$ for $x \neq y$ in $[a, c]$ and $t \geq t_o$. That is, $g(x) - g(y) \in (x - y)tN$. Given N and t_o as above, $g(x) - g(y) \in N$ for $|x - y| < \frac{1}{t_o}$. That is, as $x \to b$ the collection $g(x)$ is a bounded Cauchy net. By quasi-completeness, define $g(b) \in V$ as the limit of the values $g(x)$. For $x \to y$ the values $g(x)$ approach $g(y)$, so this extension of g is continuous on $[a, c]$. ///

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8. *Proof that weak holomorphy implies continuity*

With the above preparation, we prove that *weak holomorphy* implies (strong) *continuity*, completing the larger proof, as another application of the fact that *weak* boundedness implies boundedness, by an argument parallel to that of the first section that weak C^1 implies C^o for vector-valued functions on $[a, b]$.

[8.1] Claim: *Weak holomorphy* implies (strong) *continuity.*

Proof: To show that weak holomorphy of f implies $f : D \to V$ is (strongly) *continuous*, without loss of generality prove continuity at $z = 0$ and suppose $f(0) = 0 \in V$. Since $\lambda \circ f$ is holomorphic for each $\lambda \in V^*$ and vanishes at 0, each function $(\lambda \circ f)(z)/z$ initially defined on a punctured disk at 0 extends to a holomorphic function on a full disk at 0. By Cauchy theory for the scalar-valued holomorphic function $z \to \frac{\lambda(f(z))}{z}$,

$$\frac{(\lambda \circ f)(z)}{z} \;=\; \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} \cdot \frac{(\lambda \circ f)(w)}{w}\, dw$$

where γ is a circle of radius $2r$ centered at 0, and $|z| < r$. With M_λ the sup of $|\lambda \circ f|$ on γ,

$$\Big|\frac{(\lambda \circ f)(z)}{z}\Big| \;\leq\; \frac{\text{length}\,\gamma}{2\pi} \cdot \frac{1}{2r - r} \cdot \frac{M_\lambda}{2r} \;=\; \frac{1}{2\pi} \cdot (2\pi \cdot 2r) \cdot \frac{1}{r} \cdot \frac{M_\lambda}{2r} \;=\; \frac{M_\lambda}{r}$$

Thus, the set of values

$$S \;=\; \Big\{\frac{f(z)}{z} : |z| \leq r\Big\}$$

is *weakly* bounded. Weak boundedness implies (strong) boundedness, so S is *bounded.* That is, given a balanced convex neighborhood N of 0 in V, there is $t_o > 0$ such that for complex w with $|w| \geq t_o$, the set S lies inside wN. Then $f(z) \in zwN$ and $f(z) \in N$ for $|z| < |w|$. As $f(0) = 0$, we have proven that, given N, for z sufficiently near 0 $f(z) - f(0) \in N$. This is (strong) continuity. ///

9. *Appendix: vector-valued power series*

We should confirm that power series with values in a quasi-complete, locally compact vectorspace V behave essentially as well as scalar-valued ones. First,

[9.1] Lemma: Let c_n be a *bounded* sequence of vectors in the locally convex, quasi-complete topological vector space V. Let z_n be a sequence of complex numbers, let $0 \le r_n$ be real numbers such that $|z_n| \le r_n$, and suppose that $\sum_n r_n < +\infty$. Then $\sum_n c_n\, z_n$ *converges* in V. Further, given a convex balanced neighborhood U of 0 in V let t be a positive real such that for all complex w with $|w| \ge t$ we have $\{c_n\} \subset tU$. Then

$$\sum_n c_n\, z_n \;\in\; \Big(\sum_n |z_n|\Big)\cdot tU \subset \Big(\sum_n r_n\Big)\cdot tU$$

Proof: For convex balanced neighborhood N of 0 in the topological vector space, with complex numbers z and w such that $|z| \le |w|$, then $zN \subset wN$, since $|z/w| \le 1$ implies $(z/w)N \subset N$, or $zN \subset wN$. Further, for an absolutely convergent series $\sum_n \alpha_n$ of complex numbers, for any n_o

$$\sum_{n\le n_o} (\alpha_n\cdot V) \;=\; \sum_{n\le n_o} (|\alpha_n|\cdot V) \;\subset\; \Big(\sum_{n\le n_o} |\alpha_n|\Big)\cdot N \;\subset\; \Big(\sum_{n<\infty} |\alpha_n|\Big)\cdot N$$

For a balanced open U containing 0, let t be large enough such that for any complex w with $|w| \ge t$ the sequence c_n is contained in wU. The previous discussion shows that

$$\sum_{m\le \ell\le n} c_\ell z_\ell \;\in\; (|z_m| + \ldots + |z_n|)\cdot tU$$

Given $\varepsilon > 0$, invoking absolute convergence, take m sufficiently large such that $|z_m| + \ldots + |z_n| < t\cdot\varepsilon$ for all $n \ge m$. Then

$$\sum_{m\le \ell\le n} c_\ell z_\ell \;\in\; t\cdot(\varepsilon/t)\cdot U \;=\; U$$

Thus, the original series is convergent. Since X is quasi-complete the limit exists in V. The last containment assertion follows from this discussion, as well. ///

[9.2] Corollary: Let c_n be a *bounded* sequence of vectors in a locally convex quasi-complete topological vector space V. Then on $|z| < 1$ the series $f(z) = \sum_n c_n z^n$ converges and gives a *holomorphic* V-valued function. That is, the function is infinitely-many-times complex-differentiable.

Proof: The lemma shows that the series expressing $f(z)$ and its apparent k^{th} derivative $\sum_n c_n \binom{n}{k} z^{n-k}$ all converge for $|z| < 1$. The usual direct proof of Abel's theorem on the differentiability of (scalar-valued) power series can be adapted to prove the infinite differentiability of the X-valued function given by this power series, as follows. Let

$$g(z) \;=\; \sum_{n\ge 0} nc_n\, z^{n-1}$$

Then

$$\frac{f(z)-f(w)}{z-w} - g(w) \;=\; \sum_{n\ge 1} c_n \left(\frac{z^n - w^n}{z-w} - nw^{n-1}\right)$$

For $n = 1$, the expression in the parentheses is 1. For $n > 1$, it is

$$\big(z^{n-1} + z^{n-2}w + \ldots + zw^{n-2} + w^{n-1}\big) - nw^{n-1}$$

$$= (z^{n-1} - w^{n-1}) + (z^{n-2}w - w^{n-1}) + \ldots + (z^2w^{n-3} - w^{n-1}) + (zw^{n-2} - w^{n-1}) + (w^{n-1} - w^{n-1})$$

$$= (z-w)\left[(z^{n-2} + \ldots + w^{n-2}) + w(z^{n-3} + \ldots + w^{n-3}) + \ldots + w^{n-3}(z+w) + w^{n-2} + 0\right]$$

$$= (z-w) \sum_{k=0}^{n-2} (k+1)\, z^{n-2-k}\, w^k$$

For $|z| \le r$ and $|w| \le r$ the latter expression is dominated by

$$|z-w| \cdot r^{n-2} \, \frac{n(n-1)}{2} < |z-w| \cdot n^2\, r^{n-2}$$

Let U be a balanced neighborhood of 0 in X, and t a sufficiently large real number such that for all complex w with $|w| \ge t$ all c_n lie in wU. For $|z| \le r < 1$ and $|w| \le r < 1$, by the lemma,

$$\frac{f(z) - f(w)}{z-w} - g(w) = (z-w) \sum_{n \ge 2} c_n \cdot \left(\sum_{k=0}^{n-2} (k+1)\, z^{n-2-k}\, w^k \right) \;\in\; (z-w) \cdot \left(\sum_n n^2\, r^{n-2} \right) \cdot tU$$

Thus, as $z \to w$, eventually $\frac{f(z)-f(w)}{z-w} - g(w)$ lies in U. ///

[9.3] Corollary: Let c_n be a sequence of vectors in a Banach space X such that for some $r > 0$ the series $\sum |c_n| \cdot r^n$ converges in X. Then for $|z| < r$ the series $f(z) = \sum c_n z^n$ converges and gives a holomorphic (infinitely-many times complex-differentiable) X-valued function. ///
