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Introduction to pseudodifferential operators

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One goal is proof of local regularity of elliptic differential operators with variable coefficients, to understand regularity of Casimir operators on Riemannian symmetric space G/K with G a semi-simple real Lie group and K maximal compact. Regularity is most memorably discussed by enlarging the class of differential operators to include *pseudodifferential* operators.

We give both the Kohn-Nirenberg and Weyl *functional calculuses* for pseudodifferential operators. Although the Kohn-Nirenberg calculus of pseudodifferential operators is more immediate and accessible, the Weyl calculus has some technical advantages. One is that in the Weyl case, operators are parametrized by functions on the Heisenberg group, with composition of operators given by convolution, and transpose by $\varphi^\vee(x) = \varphi(x^{-1})$. In contrast, expression of transposes in the Kohn-Nirenberg setting requires a computation, and composition requires an extension of the notion of pseudodifferential operator.

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1. *Kohn-Nirenberg calculus, simplest symbol classes S^m*

Fourier transform converts *multiplication* by x to *differentiation* $\partial/\partial x$, which rewrites constant-coefficient differential operators to integrals against polynomials. Enlarging the class of functions beyond polynomials, the *multiplier operator* attached to a function φ is

$$f \longrightarrow \int_{\mathbb{R}} e^{2\pi i x\xi}\, \varphi(\xi)\cdot \widehat{f}(\xi)\, d\xi$$

That is, with Fourier transform $\mathscr{F}$,

$$\text{multiplier operator attached to } \varphi \;=\; \mathscr{F}^{-1}\circ \text{multiplication by } \varphi \circ \mathscr{F}$$

Extending this in another fashion gives variable-coefficient differential operators *and* sufficiently general operators to discuss solvability and regularity of partial differential equations. The fundamental complication is that the non-commutativity of variable-coefficient differential operators cannot easily be reflected in the commutative point-wise multiplication of functions, nor in convolution on $\mathbb{R}^n$. Nevertheless, other aspects of this set-up are suggestive and encouraging.

With $x, \xi \in \mathbb{R}^n$, for $p(x,\xi) = \sum_{|\alpha|\le k} c_\alpha(x)\cdot(-2\pi i\xi)^\alpha$, the operator $\mathrm{OP}(p) = \mathrm{OP}_{KN}(p)$ is

$$\mathrm{OP}(p)(f)(x) \;=\; p(x,D)f(x) \;=\; \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x}\, p(x,\xi)\, \widehat{f}(\xi)\, d\xi \;=\; \sum_{|\alpha|\le k} c_\alpha(x)\, \partial^\alpha f(x)$$

This is a variable-coefficient differential operator. We can attempt to extend $p \to \mathrm{OP}(p)$ from polynomial functions of ξ to various broader classes, by the same formula in terms of Fourier transform: the *pseudo-differential operator* $\mathrm{OP}(p) = \mathrm{OP}_{KN}(p) = p(x,D)$ attached to the *symbol* $p(x,\xi)$ in the Kohn-Nirenberg calculus is

$$\mathrm{OP}(p)(f)(x) \;=\; p(x,D)f(x) \;=\; \int_{\mathbb{R}^n} e^{2\pi i\xi\cdot x}\, p(x,\xi)\, \widehat{f}(\xi)\, d\xi$$

The notation $p(x, D)$ is suggested by the conversion of multiplication by ξ_j to application of $\partial/\partial x_j$ by Fourier transform.

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2. *Schwartz kernels in the Kohn-Nirenberg setting*

Schwartz' Kernel Theorem is that every continuous linear $T : \mathcal{D}(\mathbb{R}^m) \to \mathcal{D}(\mathbb{R}^n)^*$ has a *Schwartz kernel* $K \in \mathcal{D}(\mathbb{R}^m \times \mathbb{R}^n)^*$, meaning that

$$T(\varphi)(\psi) \;=\; K(\varphi \otimes \psi)$$

Similarly, every continuous linear $T : \mathscr{S}(\mathbb{R}^m) \to \mathscr{S}(\mathbb{R}^n)^*$ has *Schwartz kernel* $K \in \mathscr{S}(\mathbb{R}^m \times \mathbb{R}^n)^*$, again meaning that

$$T(\varphi)(\psi) \;=\; K(\varphi \otimes \psi)$$

Very nice Schwartz kernels K, for example in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, unsurprisingly correspond to very nice (if uninteresting) operators.

[2.1] Claim: For $p \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, the Schwartz Kernel K of $p(x, D) = \mathrm{OP}_{KN}(p)$ is

$$\mathscr{F}_2^{-1} p(x, x - t)$$

where $\mathscr{F}_2$ is Fourier transform in the second argument. Thus, the pseudodifferential operators given by $p \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ give exactly the operators $\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ with kernels in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof: By direct computation, for $\varphi \in \mathscr{S}(\mathbb{R}^m)$ and $\psi \in \mathscr{S}(\mathbb{R}^n)^*$,

$$p(x,D)(\varphi)(\psi) \;=\; \int_{\mathbb{R}^n} \psi(x) \Big(\int_{\mathbb{R}^m} e^{2\pi i \xi \cdot x}\, p(x,\xi)\, \widehat{\varphi}(\xi)\, d\xi \Big)\, dx \;=\; \int\!\!\int\!\!\int \psi(x)\, e^{2\pi i \xi \cdot x}\, p(x,\xi)\, e^{-2\pi i \xi \cdot t}\, \varphi(t)$$

$$= \int\!\!\int\!\!\int \psi(x)\, \varphi(t) \cdot e^{2\pi i \xi \cdot (x-t)}\, p(x,\xi) \;=\; \int\!\!\int \psi(x)\, \varphi(t) \cdot \mathscr{F}_2^{-1} p(x, x-t)$$

(Partial) Fourier transform is an automorphism of $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, as is a non-singular change of variables, giving the result. ///

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3. *The simplest symbol classes*

Traditional terminology is that various classes of functions $p(x, \xi)$ are *symbol classes*. The simplest symbol classes are

$$S^m \;=\; S^m(\Omega) \;=\; \{p \in C^\infty(\Omega \times \mathbb{R}^n) : \forall\, \text{compact}; E \subset \Omega,\, \forall \alpha, \beta,\; |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \ll_{E,\alpha,\beta} (1 + |\xi|)^{m - |\beta|}\}$$

This is the simplest example within a larger family of symbol classes studied by Hörmander: $S^m = S^m_{1,0}$ where

$$S^m_{\rho,\delta} \;=\; \{p \in C^\infty(\Omega \times \mathbb{R}^n) : \forall\, \text{compact}\ E \subset \Omega,\, \forall \alpha, \beta,\; |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \ll_{E,\alpha,\beta} (1 + |\xi|)^{m - \delta|\alpha| - \rho|\beta|}\}$$

[3.1] Remark: In more abstract situations, the space $\Omega \times \mathbb{R}^n$ would be the *tangent bundle* over Ω.

The basic result is

[3.2] Theorem: The operators $\mathrm{OP}(p) = p(x, D)$ for $p \in S^m$ are continuous maps $\mathcal{D}(\Omega) \to \mathcal{E}(\Omega)$, which extend to continuous maps $\mathcal{E}^*(\Omega) \to \mathcal{D}^*(\Omega)$.

Proof: [... iou ...] ///

[3.3] **Remark:** The continuity of the extension requires knowing that the *transpose* $p(x,D)^\top : \mathcal{D} \to \mathcal{E}$ is a pseudodifferential operator, which is not obvious in the Kohn-Nirenberg context.

[3.4] **Remark:** Naturally, a parametrization of operators by functions is most useful when composition of operators is reflected directly in some sort of composition of the parametrizing functions. A fundamental complaint one might have about the Kohn-Nirenberg parametrization is that the non-commutativity of multiplication operators and derivative operators is not well reflected in the *commutative* convolution of functions on $\mathbb{R}^n$. The Weyl parametrization (below) has the virtue of overcoming this particular difficulty.

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4. *The Schrödinger model*

The two operators

$$D = \frac{d}{dx} \qquad\qquad X = \text{multiplication by } x$$

on differentiable functions on $\mathbb{R}$ do not commute, and this is *not* a pathology. Their commutator is

$$[D,X] \;=\; D\,X - X\,D \;=\; 1$$

and the commutator *does* commute with both of them. Further, neither of these two operators stabilizes $L^2(\mathbb{R})$, although multiplication by x has an obvious pointwise sense, and $\frac{d}{dx}$ has a distributional sense on L^2 functions, since these are in L^1_{loc}.

It is non-trivial to guess a good normalization, but we can make inferences. With Fourier transform $\mathscr{F}$ normalized by

$$\mathscr{F} f(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i \xi x}\, f(x)\, dx$$

for f in the Schwartz space $\mathscr{S}$

$$D\big(\mathscr{F} f\big) \;=\; (-2\pi i)\mathscr{F}(Xf)$$

That is, we find a normalization constant of $\pm 2\pi i$ in the relation between D and X. Further, we want these operators to occur in the (complexified) image of a representation of a Lie *algebra* attached to a *unitary* Lie *group* representation. Thus, the image of the (real) Lie algebra should be *skew-hermitian* operators. Integration by parts shows that $\frac{\partial}{\partial x}$ is already skew, but multiplication by x needs a factor of $\pm i$. Thus, we might **renormalize** by taking

$$D = \frac{1}{2\pi}\frac{\partial}{\partial x} \qquad\qquad X = \text{multiplication by } ix$$

and then the commutator is

$$[D,X] \;=\; DX - XD \;=\; \frac{i}{2\pi}$$

The simplest Lie algebra whose representation theory can reflect this non-commutativity is the **Heisenberg** Lie algebra

$$\mathfrak{h} \;=\; \{\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}\}$$

which has a Lie algebra representation σ_{alg}

$$\sigma_{\text{alg}} : \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow D \qquad \sigma_{\text{alg}} : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow X \qquad \sigma_{\text{alg}} : \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \frac{i}{2\pi}$$

The fact that the Lie bracket is preserved is a direct computation. We want an associated unitary Lie *group* representation $\sigma_{\rm gp}$ of the Heisenberg *group*

$$H = \{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}\}$$

and the standard idea would be to *exponentiate* to define $\sigma_{\rm gp}$ on H by

$$\sigma_{\rm gp}(e^{2\pi\alpha}) = \exp(\sigma_{\rm alg}\alpha) \qquad \text{(for } \alpha \in \mathfrak{h}\text{)}$$

where the *first* exponentiation $\alpha \to e^{2\pi\alpha}$ is exponentiation of matrices, namely

$$e^A = 1 + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots$$

but where the *second* exponentiation, that of endomorphisms of $L^2(\mathbb{R})$, is not assured to make sense. The 2π in the exponent is in anticipation of compatibility with our normalization of Fourier transforms.

Exponentiate $\sigma_{\rm alg}$ to a group representation $\sigma_{\rm gp}$ as follows. For $a, b \in \mathbb{R}$, the effect of exponentiated $aD + bX$ on a function $f \in L^2(\mathbb{R})$ is *characterized* in a manner that should determine it uniquely, provided it exists. Let

$$g(x,t) = \exp\big(t(aD + bX)\big) \cdot f(x) \qquad \text{(for } t \text{ real)}$$

The characterization of exponentiation is

$$\frac{\partial}{\partial t}\, g(x,t) = 2\pi(aD + bX) \cdot g(x,t) \qquad \text{and} \qquad g(x,0) = f(x)$$

That is,

$$\frac{\partial}{\partial t}\, g(x,t) = 2\pi(a \cdot \frac{1}{2\pi}\frac{\partial}{\partial x} + b \cdot ix)\, g(x,t) \qquad \text{and} \qquad g(x,0) = f(x)$$

This differential equation with boundary condition can be solved directly. Rearrange to

$$(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x})\, g(x,t) = 2\pi ibx \cdot g(x,t)$$

The left-hand side is a directional derivative: fix x and let $h(t) = g(x - at, t)$, so the equation becomes

$$\frac{d}{dt}\, h(t) = 2\pi ib(x - at) \cdot h(t)$$

or

$$\frac{d}{dt}\Big(\log\, h(t)\Big) = 2\pi ib(x - at)$$

Thus,

$$g(x - at, t) = e^{2\pi ib(xt - \frac{at^2}{2})} \cdot (\text{constant depending on } x)$$

The condition $g(x,0) = f(x)$ gives

$$g(x - at, t) = e^{2\pi ib(xt - \frac{at^2}{2})} \cdot f(x)$$

Replace x by $x + at$

$$g(x,t) = e^{2\pi ib(xt + \frac{at^2}{2})} \cdot f(x + at)$$

At $t = 1$

$$\exp(aD + bX) \cdot f(x) = e^{2\pi ib(x + \frac{a}{2})} \cdot f(x + a)$$

That is,

$$\sigma_{\rm gp}\Big(\exp\begin{pmatrix} 0 & a & 0 \\ & 0 & b \\ & & 0 \end{pmatrix}\Big)\cdot f(x) \;=\; \exp(aD+bX)\cdot f(x) \;=\; e^{2\pi i b(x+\frac{a}{2})}\cdot f(x+a)$$

Including the center gives

$$\sigma_{\rm gp}\Big(\exp\begin{pmatrix} 0 & a & c \\ & 0 & b \\ & & 0 \end{pmatrix}\Big)\cdot f(x) \;=\; \exp\big(aD+bX+c\frac{i}{2\pi}\big)\cdot f(x) \;=\; e^{2\pi i b(x+\frac{a}{2})+ic}\cdot f(x+a)$$

This is the *Schrödinger model* on $L^2(\mathbb{R})$ of the (unique [1]) infinite-dimensional unitary representation of the Heisenberg group with character

$$\omega \;:\; \begin{pmatrix} 1 & 0 & c \\ & 1 & 0 \\ & & 1 \end{pmatrix} \longrightarrow e^{ic}$$

on the center Z of H. For the moment, we do not need *uniqueness* of this representation, only *existence*.

5. *Weyl's symbol calculus: operators from functions*

An association of *operators* to *functions* is a *functional calculus*. Except for the uninteresting case that the operators *commute*, the multiplication on functions cannot be the (commutative) pointwise multiplication. *Convolution* on a form of the Heisenberg group is sufficiently non-commutative to reflection composition of (pseudo-) differential operators.

For *any* representation τ of of a unimodular group G on a topological vector space V, and for suitable functions η, φ on G, the integrated actions

$$\varphi\cdot v \;=\; \int_G \varphi(g)\; g\cdot v\; dg$$

compose by

$$\eta\cdot(\varphi\cdot v) \;=\; (\eta*\varphi)\cdot v$$

where $*$ is the standard convolution determined completely by the previous relation. Formulaically, that relation determines it as

$$(\eta*\varphi)(x) \;=\; \int_G \eta(xg^{-1})\,\varphi(g)\; dg$$

Thus, for a function φ on H, the associated action of φ in the Schrödinger model on $f\in L^2(\mathbb{R})$ is

$$(\varphi\cdot f)(x) \;=\; \int_H \varphi(h)\; h\cdot f(x)\,dh \;=\; \int_H \varphi(h)\;\big(\sigma_{\rm gp}(h)\cdot f\big)(x)\,dh \;=\; \int_H \varphi(h)\; e^{2\pi i b(x+\frac{a}{2})+ci} f(x+a)\,da\,db\,dc$$

Here there is a complication: the center Z of H acts by the character ω, and the integral

$$\int_Z \omega(z)\,dz \;=\; \int_{-\infty}^{\infty} e^{ic}\; dc$$

[1] The uniqueness is the Stone-vonNeumann theorem.

diverges, so the function φ on H cannot be *constant* along the center for the integral for $\varphi \cdot f$ to behave properly. To remedy this minor flaw, note that $c \to e^{ic}$ factors through the quotient $\mathbb{R}/2\pi\mathbb{Z}$, so the group representation above factors through the quotient group

$$H^{\mathrm{red}} = H \Big/ \begin{pmatrix} 1 & 0 & 2\pi\mathbb{Z} \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

called the **reduced Heisenberg group**. The reduced Heisenberg group has the same Lie algebra $\mathfrak{h}$, but smaller center

$$Z^{\mathrm{red}} = Z \Big/ \begin{pmatrix} 1 & 0 & 2\pi\mathbb{Z} \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

Pretending some foresight, we give the reduced center $\mathbb{Z}^{\mathrm{red}}$ total measure 1, rather than the more natural 2π, for later simplification.

Since this Schödinger representation depends so trivially on the center, we can specify functions on H^{red} or H by specifying functions φ on the linear complement

$$V = \begin{pmatrix} 0 & * & 0 \\ & 0 & * \\ & & 0 \end{pmatrix} \subset \mathfrak{h}$$

to $\mathfrak{z}$ in $\mathfrak{h}$, and then writing

$$\varphi\big(\exp 2\pi(v+z)\big) = \omega^{-1}(\exp 2\pi z) \cdot \varphi(v) \qquad\qquad (\text{for } v \in V \text{ and } z \in \mathfrak{z})$$

where the ω^{-1} is chosen to cancel the ω in the Schrödinger representation. Then we may write

$$\varphi\Big(\exp 2\pi \begin{pmatrix} 0 & a & 0 \\ & 0 & b \\ & & 0 \end{pmatrix}\Big) = \varphi\begin{pmatrix} 0 & a & 0 \\ & 0 & b \\ & & 0 \end{pmatrix} = \varphi(a,b)$$

Extend this notation to allow reference to the center by

$$\varphi(a,b,c) = e^{-ic} \cdot \varphi\Big(\exp 2\pi \begin{pmatrix} 0 & a & c \\ & 0 & b \\ & & 0 \end{pmatrix}\Big) = e^{-ic} \cdot \varphi\begin{pmatrix} 0 & a & c \\ & 0 & b \\ & & 0 \end{pmatrix}$$

[5.1] Remark: When the identification of H/Z and $\mathfrak{h}/\mathfrak{z}$ with $\mathbb{R}^2$ is exaggerated, the naturality of the convolution product on H^{red} is thereby neglected.

6. *Examples of operators for Weyl calculus*

In the Schrödinger representation of H^{red} on $L^2(\mathbb{R})$, functions φ on the reduced Heisenberg group H^{red} produce recognizable operators $f \to \varphi \cdot f = \text{OP}_W(\varphi)f$.

[6.1] To get an idea, try $\varphi(a,b) = 1$. Keeping in mind that $\mathbb{Z}^{\text{red}}$ has total measure 1, not 2π,

$$\text{OP}_W(\varphi)(f)(x) \;=\; (\varphi{\cdot}f)(x) \;=\; \int_{H^{\text{red}}} \varphi(h)\; (h{\cdot}f)(x)\; dh \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}/2\pi\mathbb{Z}} e^{-ic}{\cdot}1{\cdot}e^{2\pi ib(x+\frac{a}{2})+ic}\; f(x{+}a)\, da\, db\, dc$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi ib(x+\frac{a}{2})}\; f(x+a)\, da\, db$$

Replacing a by $a - x$ and integrating in a gives

$$(\varphi \cdot f)(x) \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi ib(\frac{x+a}{2})}\; f(a)\, da\, db \;=\; \int_{\mathbb{R}} e^{2\pi ib\frac{x}{2}}\; \widehat{f}(-b/2)\, db$$

Replacing b by $2b$ gives

$$(\varphi \cdot f)(x) \;=\; 2\int_{\mathbb{R}} e^{2\pi ibx}\; \widehat{f}(-b)\, db \;=\; 2\cdot f(-x)$$

by Fourier inversion.

[6.2] Next, try

$$\varphi(a,b) \;=\; a$$

A similar computation, integrating first in b, gives

$$(\varphi \cdot f)(x) \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}/2\pi\mathbb{Z}} e^{-ic}\cdot a\cdot e^{2\pi ib(x+\frac{a}{2})+ic}\; f(x+a)\, da\, db\, dc \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} a\cdot e^{2\pi ib(x+\frac{a}{2})}\; f(x+a)\, da\, db$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} (a-x)\cdot e^{2\pi ib(x+a)/2}\; f(a)\, da\, db \;=\; 2\int_{\mathbb{R}}\int_{\mathbb{R}} (a-x)\cdot e^{2\pi ib(x+a)}\; f(a)\, da\, db$$

$$= 2\int_{\mathbb{R}} (a-x)\cdot\delta(x+a)\; f(a)\, da \;=\; 2\cdot(-x-x)\, f(-x) \;=\; -4\pi\cdot xf(-x)$$

That is, up to the change of sign in the argument to f, $\text{OP}(\varphi)$ is a *multiplication* operator.

[6.3] Now try

$$\varphi(a,b) \;=\; b$$

A similar computation:

$$(\varphi \cdot f)(x) \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}/2\pi\mathbb{Z}} e^{-ic}\cdot b\cdot e^{2\pi ib(x+\frac{a}{2})+ic}\; f(x+a)\; da\, db\, dc \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} b\cdot e^{2\pi ib(x+\frac{a}{2})}\; f(x+a)\; da\, db$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} b\cdot e^{2\pi ib(\frac{x+a}{2})}\; f(a)\; da\, db \;=\; \frac{-i}{\pi}\frac{\partial}{\partial x}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi ib(\frac{x+a}{2})}\; f(a)\; da\, db$$

$$= \frac{-i}{\pi}\frac{\partial}{\partial x}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{2\pi ib(x+a)}\; f(a)\; da\, db \;=\; \frac{-i}{\pi}\frac{\partial}{\partial x}\int_{\mathbb{R}} \delta(x+a)\, f(a)\, da \;=\; \frac{-i}{\pi}\frac{\partial}{\partial x} f(-x)$$

That is, we can obtain constant coefficient *differential* operators.

Some of the details above are mere artifaces of coordinate choices. In fact, a significant advantage here is the *intrinsic* sense of the map from functions to operators.

7. *Computing Schwartz' kernels*

The possibilities and normalizations for operators appearing as $\sigma_{\rm gp}(\varphi)$ are best understood by determining the *kernel* K_φ of the operator attached to φ, in the sense of Schwartz

$$(\varphi\cdot f)(x) \;=\; \int_{\mathbb{R}} K_\varphi(x,a)\cdot f(a)\,da$$

Direct computation shows that Schwartz functions $\mathscr{S}(V)$ on the complementary subspace V to $\mathfrak{z}$ inside $\mathfrak{h}$ biject to kernels in $\mathscr{S}(\mathbb{R}\times\mathbb{R})$.

We would expect to extend by continuity to a bijection of tempered distributions on V to kernels in $\mathscr{S}(\mathbb{R}\times\mathbb{R})^*$. Schwartz' kernel theorem says that every operator $\mathscr{S}(\mathbb{R})\to\mathscr{S}(\mathbb{R})^*$ is given by an element of $\mathscr{S}(\mathbb{R}\times\mathbb{R})^*$, thus, by $\sigma_{\rm gp}(\varphi)$ for some $\varphi\in\mathscr{S}(V)^*$.

[7.1] Computing the kernel A direct computation gives the kernel: using the notational convention above for φ a function on $V\subset\mathfrak{h}$,

$$(\varphi\cdot f)(x) \;=\; \int_{H^{\rm red}} \varphi(h)\cdot\big(h\cdot f\big)(x)\,dh \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}/2\pi\mathbb{Z}} e^{-ic}\cdot\varphi(a,b)\cdot e^{2\pi ib(x+\frac{a}{2})+ic}\,f(x+a)\,da\,db\,dc$$

$$=\; \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(a,b)\cdot e^{2\pi ib(x+\frac{a}{2})}\,f(x+a)\,da\,db \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(a-x,b)\cdot e^{2\pi ib(\frac{x+a}{2})}\,f(a)\,da\,db$$

$$=\; \int_{\mathbb{R}} \mathscr{F}_2^{-1}\,\varphi\Big(a-x,\frac{x+a}{2}\Big)\,f(a)\,da$$

where $\mathscr{F}_2$ is Fourier transform in the second argument. Thus, the kernel K_φ for the operator $\sigma_{\rm gp}(\varphi)$ is

$$K_\varphi(x,a) \;=\; \mathscr{F}_2^{-1}\varphi\Big(a-x,\frac{x+a}{2}\Big)$$

(Partial) Fourier transform and linear changes of coordinates are automorphisms of the Schwartz space.

[7.2] Examples As a trivial confirmation, consider the identity map 1 on $\mathscr{S}$, which has kernel

$$K_1(x,a) \;=\; \delta(x-a) \;\in\; \mathscr{S}'(\mathbb{R}\times\mathbb{R})$$

To find $\varphi\in\mathscr{S}'(V)$ producing this kernel, solve

$$\mathscr{F}_2^{-1}\,\varphi\Big(a-x,\frac{x+a}{2}\Big) \;=\; \delta(a-x) \;=\; \delta(a-x)\;\otimes\;1\Big(\frac{x+a}{2}\Big)$$

Then

$$\varphi\Big(a-x,\frac{x+a}{2}\Big) \;=\; \delta(a-x)\otimes\delta\Big(\frac{x+a}{2}\Big)$$

8. *Appendix: nuclearity of $\mathscr{S}$ and of $\mathcal{D}$*

For us, *nuclear* topological vectorspaces M,N have the property that there is a topological vectorspace $M\otimes N$ behaving as a genuine tensor product in a suitable category of locally convex topological vectorspaces. That is, there is a continuous bilinear $M\times N\to M\otimes N$ such that, for every continuous bilinear $\beta: M\times N\to X$ to a topological vector space X in the category there is a unique linear $B : M \otimes N \to X$ giving a commutative diagram

$$\begin{tikzcd} M \otimes N \arrow[dr, dotted, "\exists!"] & \\ M \times N \arrow[u] \arrow[r, dashed, "\forall"] & X \end{tikzcd}$$

To build up a class of such spaces, begin with *nuclear Fréchet* spaces: countable (projective) limits of Hilbert spaces with *Hilbert-Schmidt* transition maps.

That $\mathcal{E}(\mathbb{T}^n)$ is nuclear Fréchet follows from two things: the Sobolev result that $H^\infty(\mathbb{T}^n) = \mathcal{E}(\mathbb{T}^n)$, and the Rellich result that $H^k(\mathbb{T}^n) \to H^{k-\ell}(\mathbb{T}^n)$ is Hilbert-Schmidt for $\ell \gg_n 1$.

That Schwartz spaces $\mathscr{S}(\mathbb{R}^n)$ are nuclear Fréchet is perhaps less obvious, since the

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seen as follows. Let $S = -\Delta + |x|^2$ be the *Hermite operator*, also known as the (Hamiltonian for) the *quantum harmonic oscillator*. Define norms on $\mathcal{D}(\mathbb{R}^n)$ by

$$|f|^2_{\mathfrak{B}^k} \;=\; \langle S^k f, f\rangle_{L^2} \qquad\qquad (\text{for } f \in \mathcal{D})$$

and let $\mathfrak{B}^k$ be the completion of $\mathcal{D}$ with respect to the $\mathfrak{B}^k$-norm. Then one shows two things: that $\mathfrak{B}^k \to \mathfrak{B}^{k-\ell}$ is Hilbert-Schmidt for $\ell \gg_n 1$, and that $\mathscr{S} = \mathfrak{B}^\infty = \lim_k \mathfrak{B}^k$. As in [Garrett 2014], for example, this proves that $\mathscr{S}$ is nuclear Fréchet.

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