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Riesz' Lemma

Paul Garrett garrett@math.umn.edu <http://www.math.umn.edu/~garrett/>

[This page is http://www.math.umn.edu/~garrett/m/fun/riesz_lemma.pdf]

This elementary inequality is often referred to as *Riesz' Inequality*, but is much less often indexed as such in the relevant texts and monographs, making it inconvenient to find by that name. It is simple and widely useful. A sample reference is [Riesz-Nagy 1952] page 218.

This little lemma is the Banach-space substitute for one aspect of *orthogonality* in Hilbert spaces. In a Hilbert space Y , given a non-dense subspace X , there is $y \in Y$ with $|y| = 1$ and $\inf_{x \in X} |x - y| = 1$, by taking y in the *orthogonal complement* to X . Of course, this device is unavailable more generally.

[0.1] Lemma: (*Riesz*) For a non-dense subspace X of a Banach space Y , given $r < 1$, there is $y \in Y$ with $|y| = 1$ and $\inf_{x \in X} |x - y| \geq r$.

Proof: Take y_1 not in the closure of X , and put $R = \inf_{x \in X} |x - y_1|$. Thus, $R > 0$. For $\varepsilon > 0$, let $x_1 \in X$ be such that $|x_1 - y_1| < R + \varepsilon$. Put $y = (y_1 - x_1)/|x_1 - y_1|$, so $|y| = 1$. And

$$\begin{aligned} \inf_{x \in X} |x - y| &= \inf_{x \in X} \left| x + \frac{x_1}{|x_1 - y_1|} - \frac{y_1}{|x_1 - y_1|} \right| = \inf_{x \in X} \left| \frac{x}{|x_1 - y_1|} + \frac{x_1}{|x_1 - y_1|} - \frac{y_1}{|x_1 - y_1|} \right| \\ &= \frac{\inf_{x \in X} |x - y_1|}{|x_1 - y_1|} \geq \frac{R}{R + \varepsilon} \end{aligned}$$

By choosing $\varepsilon > 0$ small, $R/(R + \varepsilon)$ can be made arbitrarily close to 1. ///

[Riesz-Nagy 1952] F. Riesz, B. Szökefalvi-Nagy, *Functional Analysis*, English translation, 1955, L. Boron, from *Lecons d'analyse fonctionelle* 1952, F. Ungar, New York, 1955.
