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Smoothing distributions

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Every locally integrable function f gives a distribution by *integrating against* it, as in

$$\varphi \longrightarrow \int_{\mathbb{R}^n} \varphi(x)\, f(x)\, dx \qquad\qquad (\text{for } \varphi \in C_c^\infty(\mathbb{R}^n))$$

Conversely, we prove here that any distribution u can be approximated arbitrarily well in the weak-*-topology by (integration against) smooth functions. [1] Further, a *sequence* of such smooth functions approaching u can be exhibited in terms of *smoothing* or *mollifying* u.

Let $g \to T_g$ be the *right regular representation* of $\mathbb{R}^n$ on test functions $f \in C_c^\infty(\mathbb{R}^n)$ by

$$(T_g f)(x) \;=\; f(x+g) \qquad\qquad (\text{for } x, g \in \mathbb{R}^n)$$

One should verify that $x \times f \to T_x f$ gives a continuous map

$$\mathbb{R}^n \times C_c^\infty(\mathbb{R}^n) \longrightarrow C_c^\infty(\mathbb{R}^n)$$

At various moments, to lighten the notation we suppress the T and write $g \cdot f$ for $T_g f$. The corresponding *adjoint* action of $\mathbb{R}^n$ on distributions u is

$$(T_g^* u)(f) \;=\; u(T_g^{-1} f)$$

One should verify that $x \times u \to x \cdot u = T_x^* u$ is a continuous map

$$\mathbb{R}^n \times C_c^\infty(\mathbb{R}^n)^* \longrightarrow C_c^\infty(\mathbb{R}^n)^*$$

Suppress the T^* and write $g \cdot u$ for $T_g^* f$ whenever this helps readability. The usual action of a function $\varphi \in C_c^o(\mathbb{R}^n)$ on distributions u is by *integrating* the group action [2]

$$T_\varphi^* u \;=\; \int_{\mathbb{R}^n} \varphi(x)\, T_x^* u\, dx \;\in\; C_c^\infty(G)^*$$

Suppressing the T^*, this is

$$\varphi \cdot u \;=\; \int_{\mathbb{R}^n} \varphi(x)\, x \cdot u\, dx \;\in\; C_c^\infty(G)^*$$

A **smooth approximate identity** on $\mathbb{R}^n$ is a sequence ψ_i of test functions on $\mathbb{R}^n$ such that

$$\begin{cases} \int_{\mathbb{R}^n} \psi_i(x)\, dx = 1 \\ \psi_i(x) \geq 0 \\ \text{supports of the } \psi_i \text{ shrink to } 0 \end{cases}$$

[1] It is also true that the smooth functions can be chosen to have *compact support*, but this is not the main point.

[2] The Gelfand-Pettis/weak distribution-valued integral exists because the space of distributions is *quasi-complete* and *locally convex* and the function $x \to x \cdot u$ is a continuous compactly-supported $C_c^\infty(\mathbb{R}^n)^*$-valued function.

The last requirement has the meaning that, for every neighborhood N of 0, there is i_o sufficiently large such that for $i \ge i_o$ the support of ψ_i is inside N.

[0.0.1] Theorem: For an approximate identity $\{\psi_i\}$ and distribution u, the distributions $\psi_i \cdot u$ go to u in the weak-*-topology on $C_c^\infty(\mathbb{R}^n)^*$, and are (integration against) the functions $x \to u(T_x^{-1}\psi_i)$, which are *smooth functions.*

Proof: First, we prove that $\psi_i \cdot u \to u$ as distributions. Let U be any *convex*[3] open neighborhood of 0 in $C_c^\infty(\mathbb{R}^n)^*$. Let N be a sufficiently small neighborhood of 0 in $\mathbb{R}^n$ such that under

$$\mathbb{R}^n \times C_c^\infty(\mathbb{R}^n)^* \longrightarrow C_c^\infty(\mathbb{R}^n)^*$$

we have

$$N \times u \longrightarrow N \cdot u \subset u + \frac{1}{2}U$$

For i sufficiently large so that the support of ψ_i is inside N, the measure $\psi_i(x)\,dx$ is a probability measure supported in N, so by Gelfand-Pettis

$$\psi_i \cdot u \in \text{closure of convex hull of image of } \psi_i(x)\, x \cdot u$$

Since $u + U' + \frac{1}{2}U$ contains the closure of the convex set $u + \frac{1}{2}U$ for any open U' containing 0 (in $C_c^\infty(\mathbb{R}^n)^*$), this shows that

$$\psi_i \cdot u \in u + U$$

This holds for all open neighborhoods U of 0, so $\psi_i \cdot u \to u$.

To prove that every $f \cdot u$ for $f \in C_c^\infty(\mathbb{R}^n)$ is (integration against) a *continuous* or *smooth* function, we first guess what that continuous function is, by telling its point-wise values. Indeed, if $u = u_\varphi$ were known to be integration against a continuous function φ, then with an approximate identity $\{\psi_i\}$

$$\lim_i u_\varphi(\psi_i) = \lim_i \int_{\mathbb{R}^n} \varphi(x)\,\psi_i(x)\,dx = \varphi(0)$$

Thus, we anticipate determining values of the alleged continuous function $f \cdot u$ by computing

$$\text{alleged value } (f \cdot u)(0) = \lim_i (f \cdot u)(\psi_i)$$

For a continuous function F on $\mathbb{R}^n$, let

$$F^\vee(x) = F(-x)$$

For for f and ψ in $C_c^\infty(\mathbb{R}^n)$, using the fundamental fact that Gelfand-Pettis integrals commute with continuous linear maps, compute

$$(f \cdot u)(\psi) = \Big(\int_{\mathbb{R}^n} f(x)\, x \cdot u\, dx\Big)(\psi) = \int_{\mathbb{R}^n} f(x)\,(x \cdot u)(\psi)\,dx$$

$$= \int_{\mathbb{R}^n} f(x)\, u(T_x^{-1} \cdot \psi)\,dx = u\Big(\int_{\mathbb{R}^n} f(x)\,(T_x^{-1} \cdot \psi)\,dx\Big) = u\Big(\int_{\mathbb{R}^n} f(-x)\,(x \cdot \psi)\,dx\Big) = u(f^\vee \cdot \psi)$$

Note that at one point we used the fuller notation to avoid the confusing notation $-x \cdot \psi$. The function $f^\vee \cdot \psi$ admits a rewriting that reverses the roles of f and ψ, namely

[3] The fact that 0 has a local basis of *convex* open neighborhoods is the local convexity of the space of distributions.

$$(f^\vee \cdot \psi)(y) \;=\; \int_{\mathbb{R}^n} f(-x)\, \psi(y+x)\, dx \;=\; \int_{\mathbb{R}^n} f(y-x)\, \psi(x)\, dx$$

$$=\; \int_{\mathbb{R}^n} f(y+x)\, \psi(-x)\, dx \;=\; \int_{\mathbb{R}^n} f(y+x)\, \psi^\vee(x)\, dx \;=\; (\psi^\vee \cdot f)(y)$$

Thus,

$$(f\cdot u)(\psi) \;=\; u(f^\vee \cdot \psi) \;=\; u(\psi^\vee \cdot f) \;=\; (\psi\cdot u)(f)$$

We already know that $\psi_i \cdot u \to u$ for an approximate identity ψ_i, so the limit exists, and has an understandable value:

$$(f\cdot u)(\psi_i) \;=\; (\psi_i \cdot u)(f) \to u(f) \;=\; \text{supposed value of } f\cdot u \text{ at } 0$$

Thus, we would guess that $f\cdot u$ should be a function with value $u(f)$ at 0. More generally, for the distribution u_φ given by integration against φ, we have

$$(T_z^* u_\varphi)(\psi_i) \;=\; u_\varphi(T_z^{-1}\psi_i) \;=\; \int_{\mathbb{R}^n} \varphi(x)\, \psi_i(x-z)\, dx \;=\; \int_{\mathbb{R}^n} \varphi(x+z)\, \psi_i(x)\, dx \;\to\; \varphi(z)$$

Then do the analogous computation to guess the values of the function $f\cdot u$ at z. First, a more elaborate version of the identity reverses the roles of test functions f and φ, namely

$$(f^\vee \cdot T_z^{-1}\psi)(y) \;=\; \int_{\mathbb{R}^n} f(-x)\, \psi(y+x-z)\, dx \;=\; \int_{\mathbb{R}^n} f(y-x-z)\, \psi(x)\, dx$$

$$=\; \int_{\mathbb{R}^n} f(y+x-z)\, \psi(-x)\, dx \;=\; \int_{\mathbb{R}^n} (T_z^{-1} f)(y+x)\, \psi^\vee(x)\, dx \;=\; (\psi^\vee \cdot T_z^{-1} f)(y)$$

The same sort of computation gives

$$(T_y^*(f\cdot u))(\psi_i) \;=\; (f\cdot u))(T_y^{-1}\psi_i) \;=\; u(f^\vee \cdot T_y^{-1}\psi_i) \;=\; u(\psi_i^\vee \cdot T_y^{-1} f)$$

$$=\; (T_y^*(\psi_i \cdot u))(f) \to (T_y^* u)(f) \;=\; u(T_y^{-1} f) \;=\; \text{supposed value of } f\cdot u \text{ at } y$$

Since $\mathbb{R}^n \times C_c^\infty(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)$ is continuous, and u is continuous, the composition

$$y \times f \;\longrightarrow\; T_y^{-1} f \;\longrightarrow\; u(T_y^{-1} f)$$

is indeed *continuous* as a function of $y \in \mathbb{R}^n$.

Now we check that the distribution $f\cdot u$ is truly given by integration against the continuous function

$$\varphi(y) \;=\; u(T_y^{-1} f)$$

that apparently gives the pointwise values of $f\cdot u$. Letting $h \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \varphi(x)\, h(x)\, dx \;=\; \int_{\mathbb{R}^n} u(T_x^{-1} f)\, h(x)\, dx \;=\; \Big(\int_{\mathbb{R}^n} h(x)\, x\cdot u\, dx\Big)(f) \;=\; (h\cdot u)(f)$$

We already computed directly that

$$(h\cdot u)(f) \;=\; u(h^\vee \cdot f) \;=\; u(f^\vee \cdot h) \;=\; (f\cdot u)(h)$$

which shows that integration against the continuous function $\varphi(y) = u(T_y^{-1} f)$ gives the distribution $f\cdot u$.

Smoothness of $\varphi(y) = u(T_y^{-1} f)$ would follow from the assertion that $y \to T_y^{-1} f$ is a smooth, $C_c^\infty(R^n)$-valued function. The latter assertion is existence of the limit

$$\lim_{t\to 0} \frac{T_{y+tX}^{-1} f - T_y^{-1} f}{t} \qquad\qquad (\text{for } X \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^n)$$

in $C_c^\infty(\mathbb{R}^n)$ for each $X \in \mathbb{R}^n$. Clearly it suffices to consider $y = 0$. Then the fact that, by design, differentiation is a continuous map of $C_c^\infty(\mathbb{R}^n)$ to itself, is the requisite *smoothness*. ///

[0.0.2] Remark: Note that the proof of the fact that $\psi_i \cdot u \to u$ did not use the specifics of the situation: the same argument applies to representations of Lie groups.

[0.0.3] Remark: Although we could have verified that the distribution $f \cdot u$ is given by integration against $u(T_x^{-1} f)$, it is worthwhile to see that one can *infer* this. That is, given the idea that $f \cdot u$ has been *smoothed*, determination of it as a classical function is straightforward.
