(*May 10, 2008*)

Uncertainty principles in Fourier analysis

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

The Heisenberg Uncertainty Principle is a theorem about Fourier transforms, once we grant a certain model of quantum mechanics. That is, there is an unavoidable mathematical mechanism that yields an inequality, which has an interpretation in physics. [1]

For suitable f on $\mathbb{R}$,

$$|f|_{L^2}^2 \;=\; \int_{\mathbb{R}} |f|^2 \;=\; -\int_{\mathbb{R}} x(f\cdot\overline{f})' \;=\; -2\mathrm{Re}\int_{\mathbb{R}} xf\overline{f}' \qquad \text{(integrating by parts)}$$

That is,

$$|f|_{L^2}^2 \;=\; \big||f|_{L^2}^2\big| \;=\; \left|\int_{\mathbb{R}} |f|^2\right| \;=\; \left|-2\mathrm{Re}\int_{\mathbb{R}} xf\overline{f}'\right| \;\le\; 2\int_{\mathbb{R}} |xf\overline{f}'|$$

Next,

$$2\int_{\mathbb{R}} |xf\cdot\overline{f}'| \;\le\; 2\cdot|xf|_{L^2}\cdot|f'|_{L^2} \qquad \text{(Cauchy-Schwarz-Bunyakowsky)}$$

Since Fourier transform is an isometry, and since Fourier transform converts derivatives to multiplications,

$$|f'|_{L^2} \;=\; |\widehat{f}'|_{L^2} \;=\; 2\pi|\xi\widehat{f}|_{L^2}$$

Thus, we obtain the **Heisenberg inequality**

$$|f|_{L^2}^2 \;\le\; 4\pi\cdot|xf|_{L^2}\cdot|\xi\widehat{f}|_{L^2}$$

More generally, a similar argument gives, for any $x_o\in\mathbb{R}$ and any $\xi_o\in\mathbb{R}$,

$$|f|_{L^2}^2 \;\le\; 4\pi\cdot|(x-x_o)f|_{L^2}\cdot|(\xi-\xi_o)\widehat{f}|_{L^2}$$

Imagining that $f(x)$ is the probability that a particle's *position* is x, and $\widehat{f}(\xi)$ is the probability that its momentum is ξ, Heisenberg's inequality gives a lower bound on how *spread out* these two probability distributions must be. The physical assumption is that position and momentum *are* related by Fourier transform.

[1] I think I first saw Heisenberg's Uncertainty Principle presented directly as a theorem about Fourier transforms in Folland's 1983 Tata Lectures on PDE.