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Uncountable coproducts of topological vector spaces

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The point of this note is that locally convex coproducts of locally convex topological vector spaces *fail* to be coproducts in the larger category of not-necessarily-locally-convex topological vector spaces.

[1.1] Locally convex coproducts

An arbitrary collection $\{V_i : i \in I\}$ of locally convex topological vector spaces V_i has a *locally convex* coproduct $\coprod V_i$, constructed as follows.

The underlying vector space is the algebraic coproduct of the V_i, that is, the collection of functions v on I such that $v(i) \in V_i$, and for all-but-finitely-many i, $v(i) = 0$. The injection $j_i : V_i \to V$ is by inserting V_i at the i^{th} spot:

$$j_i(v)(i') \;=\; \begin{cases} v & (\text{for } i' = i) \\ \\ 0 & (\text{otherwise}) \end{cases} \qquad\qquad (\text{for } v \in V_i)$$

The topology [1] has a local basis at 0 consisting of sets of the form

$$\text{convex hull}\,\Big(\sum_{i \in I} j_i(U_i)\Big) \qquad\qquad (\text{for } U_i \text{ open in } V_i)$$

where the sum denotes all *finite* sums of elements from the images $j_i(U_i)$.

Verifying that this constructs a locally convex coproduct: given a family $f_i : V_i \to X$ of continuous linear maps to locally convex X, the corresponding $f : \coprod_{i \in I} V_i \to X$ is

$$f(v) \;=\; \sum_{i \in I} f_i(v(i)) \qquad\qquad (\text{for } v \in \textstyle\coprod_{i \in I} V_i)$$

[1.2] $\ell^p(I)$ spaces with $0 < p < 1$

With $0 < p < 1$, for an infinite index set I, the topological vector space

$$\ell^p(I) \;=\; \{\{x_i \in \mathbb{C}\} \;:\; \sum_{i \in I} |x_i|^p \;<\; \infty\}$$

is definitely *not locally convex* with the topology given by the metric $d(x, y) = |x - y|_p$ coming from

$$|x|_p \;=\; \sum_i |x_i|^p \qquad\qquad (\text{for } 0 < p < 1 \text{ no } p^{th} \text{ root!})$$

It is *complete* with respect to this metric. Note that $|x|_p$ fails to be a *norm* by failing to be homogeneous of degree 1. The failure of local convexity is proven as follows.

Local convexity would require that the convex hull of the δ-ball at 0 be contained in some r-ball. That is, local convexity would require that, given δ, there is some r such that

$$\Big|\frac{1}{n}\cdot(\delta, 0, \ldots) + \ldots + \frac{1}{n}\cdot(\underbrace{0, \ldots, 0, \delta}_{n}, 0, \ldots)\Big|_p \;=\; \Big(\frac{\delta}{n}\Big)^p + \ldots + \Big(\frac{\delta}{n}\Big)^p \;<\; r \qquad\qquad (\text{for } n = 1, 2, 3, \ldots)$$

[1] There exists a tradition to call this the *diamond* topology. It might also make sense to call it the *octagon* topology, or *kite* topology, depending on one's sensibilities.

That is, local convexity would require that, given δ, there is r such that

$$n^{1-p} \; < \; \frac{r}{\delta^p} \qquad\qquad (\text{for } n = 1, 2, 3, \ldots)$$

This is impossible because $0 < p < 1$. ///

To prove the triangle inequality, it suffices to prove that

$$(x+y)^p \; < \; x^p + y^p \qquad\qquad (\text{for } 0 < p < 1 \text{ and } x, y \geq 0)$$

Take $x \geq y$. By the mean value theorem,

$$(x+y)^p \; \leq \; x^p + p\xi^{p-1} y \qquad\qquad (\text{for some } x \leq \xi \leq x+y)$$

and

$$\begin{aligned} x^p + p\xi^{p-1} y \; &\leq \; x^p + p x^{p-1} y \; \leq \; x^p + p y^{p-1} y \; = \; x^p + p y^p \\ &\leq \; x^p + y^p \qquad\qquad (\text{since } p - 1 < 0 \text{ and } \xi \geq x \geq y) \end{aligned}$$

This proves the triangle inequality for $0 < p < 1$. ///

[1.3] Disparities between coproducts

Without directly considering the not-necessarily-locally-convex coproduct, we can show that the locally convex coproduct of an *uncountable* number of locally convex topological vector spaces *fails* to be a coproduct in the larger category of not-necessarily-locally-convex topological vector spaces. For example, we show that a locally convex coproduct $\coprod_{i \in I} \mathbb{C}_i$ of copies $\mathbb{C}_i \approx \mathbb{C}$ of complex lines $\mathbb{C}$ admits no continuous linear maps f to $\ell^p(I)$ for $0 < p < 1$ compatible with the maps

$$f_i(z)(i') \; = \; \begin{cases} z & (\text{for } i' = i) \\ \\ 0 & (\text{otherwise}) \end{cases} \qquad\qquad (\text{mapping } \mathbb{C}_i \to \ell^p(I))$$

Proof: The intersection of $j_i(\mathbb{C}_i)$ with an open neighborhood N of 0 in the locally convex coproduct contains the image $j_i(\delta_i)$ for some $\delta_i > 0$. Using the uncountability of the index set I, there is $\delta > 0$ such that there are infinitely-many indices $i_1, i_2, \ldots$ in I such that $\delta_{i_k} \geq \delta$. For $f(N)$ to lie inside the a ball B_r of radius r at 0 in $\ell^p(I)$, every convex combination

$$\frac{j_{i_1}(\delta_{i_1})}{n} + \ldots + \frac{j_{i_n}(\delta_{i_n})}{n}$$

would lie in B_r. This would require

$$\frac{\delta_{i_1}^p}{n^p} + \ldots + \frac{\delta_{i_n}^p}{n^p} \; < \; r$$

which would imply

$$n \cdot \frac{\delta^p}{n^p} \; < \; r$$

or

$$n^{1-p} \; < \; \frac{r}{\delta^p}$$

The right hand side is fixed, and the left-hand side goes to $+\infty$ as $n \to +\infty$, because $p < 1$, which is impossible. ///
