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von Neumann density theorem

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Theorem: Let V be a Hilbert space, and A a $*$-stable $\mathbf{C}$-algebra of continuous linear operators on V containing the constants. Then for any collection $\{v_n\}$ of vectors in V with $\sum_n |v_n|^2 < \infty$, for all $\varepsilon > 0$ and for all T in the double commutant A'' of A in $\mathrm{End}_{\mathbf{C}}(V)$, there is $x \in A$ such that

$$\sum_n |(T-x)v_n|^2 < \varepsilon$$

Remark: This asserts that A is dense in A'' in the **ultra-strong topology** on operators on V, given by seminorms $p_{\{v_n\}}$ for vectors v_n with $\sum_n |v_n|^2 < \infty$, defined by

$$p_{\{v_n\}}(T)^2 = \sum_n |Tv_n|^2$$

Recall that the **strong** topology is given by the seminorms p_v for $v \in V$ defined by

$$p_v(T) = |Tv|$$

V.S. Varadarajan has remarked that A. Weil first noted the appearance of the strong operator topology in such a circumstance. See J. Dixmier's *von Neumann algebras* for more information.

Proof: First, we claim that for each $v \in V$

$$A''v \subset \overline{Av}$$

where the overbar denotes closure in V. The closure $\overline{Av}$ is A-stable, and A is $*$-stable so the orthogonal complement to $\overline{Av}$ is also A-stable. Thus, the orthogonal projection $P : V \to \overline{Av}$ to it commutes with every element of A: for $v \in V$

$$PTv = PT(Pv + (1_V - P)v) = TPv + 0 = TPv$$

since PV and $(1_V - P)V$ are T-stable. Then $T \in A''$ commutes with P. Thus, as $1_V \in A$,

$$Tv = T \cdot 1_V v = T(Pv) = P(Tv) \in \overline{Av}$$

So there is a sequence $a_1, a_2, \ldots$ in A such that $a_n v \to Tv$. Thus A is dense in A'' in the strong operator topology. The argument can be enhanced to prove density of A in A'' in the ultra-strong topology, as follows.

Let W be the Hilbert space of sequences $\{v_n : n \geq 1\}$ with $\sum_n |v_n|^2 < \infty$. Let $x \in A$ act diagonally on W, by

$$x^\Delta\{v_n\} = \{xv_n\}$$

Let $p_n : W \to V$ be the orthogonal projection to the n^{th} component, and $i_n : V \to W$ the imbedding of V at the n^{th} component. For $S \in \mathrm{End}_{\mathbf{C}}(W)$ commuting with the diagonal action A^Δ of A, for $x \in A$

$$(p_m S x^\Delta)\{v_n\} = (p_m x^\Delta S)\{v_n\} = (x p_m S)\{v_n\}$$

And for $v \in V$

$$(S i_n x)v = (S x^\Delta i_n)v = (x^\Delta S i_n)v$$

Thus, for all indices m, n,

$$p_m S i_n \in A'$$

For $T \in A''$, we claim that the diagonal action

$$T^\Delta\{v_n\} = \{Tv_n\}$$

lies in the double commutant $(A^\Delta)''$. It suffices to prove that for S in the commutant $(A^\Delta)'$

$$p_m(ST^\Delta - T^\Delta S)i_n = 0 \in A'$$

for all indices m, n. Having already noted that $p_n S i_m$ is in A',

$$p_n S T^\Delta i_m = (p_n S i_m)T = T(p_n S i_m) = p_n T^\Delta S i_m$$

as desired. Thus, by the first part of this proof, T^Δ can be approximated in the strong topology on W by elements of A^Δ, so T can be approximated in the ultra-strong topology on V by elements of A. ///
