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Young's inequality

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The *numerical* Young's inequality is

$$ab \;\le\; \frac{a^p}{p} + \frac{b^q}{q} \qquad\qquad (\text{for } a, b > 0 \text{ and } \frac{1}{p} + \frac{1}{q} = 1,\ p, q > 0)$$

The proof (below) applies the convexity of *logarithm* to judiciously chosen inputs. It is so straightforward that its proof is often omitted. Note that the $p = q = 2$ case has an even simpler proof:

$$0 \;\le\; (a-b)^2 \;=\; a^2 - 2ab + b^2$$

rearranges to

$$ab \;\le\; \frac{a^2}{2} + \frac{b^2}{2} \qquad\qquad (\text{for } a, b \ge 0)$$

Proof: Convexity of logarithm is that its graph lies *above* the line segment connecting two points on the graph:

$$t \cdot \log x + (1-t) \cdot \log y \;\le\; \log(t \cdot x + (1-t) \cdot y) \qquad\qquad (\text{for } x, y > 0 \text{ and } 0 \le t \le 1)$$

Thus,

$$\frac{1}{p} \log a^p + \frac{1}{q} \log b^q \;\le\; \log\Big(\frac{a^p}{p} + \frac{b^q}{q}\Big)$$

That is,

$$\log a + \log b \;\le\; \log\Big(\frac{a^p}{p} + \frac{b^q}{q}\Big)$$

Exponentiating gives the inequality. ///

Part of the point is to mention the original papers [Young 1912a] and [Young 1912b]. The result has been so well assimilated that already the extensive bibliography of [Riesz-Nagy 1952] did not list these papers, although others of Young's did appear.

[Riesz-Nagy 1952] F. Riesz, B. Szökefalvi.-Nagy, *Functional Analysis*, English translation, 1955, L. Boron, from *Lecons d'analyse fonctionelle* 1952, F. Ungar, New York, 1955

[wiki/Young's_inequality 2012] http://en.wikipedia.org/wiki/Young's_inequality, retrieved Feb 29, 2012

[Young 1912a] W.C. Young, *On classes of summable functions and their Fourier series*, Proc. Royal Soc. Lond. **87** (1912), 225-229.

[Young 1912b] W.C. Young, *On the multiplication of successions of Fourier constants*, Proc. Royal Soc. Lond. **87** (1912), 331-339.
