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00 Introduction: homological and commutative algebra

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A fundamental idea emphasized by *category theory* is *characterization* of objects by their *interactions* with other objects, rather than *constructions* and internal details. [1]

More generally:

Sufficiently many good *examples* should make formal definitions nearly unnecessary, and should make most basic results *obvious*.

A good methodological and technical viewpoint should enable not merely *verification* of interesting results, but should assist *discovery*.

Phenomena are more interesting than symbols.

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1. *Some recurring themes*

There are many threads here, touched at various levels of sophistication, interacting in many ways. The following are *some* of the recurring ideas:

Naive category theory has the immediate feature of characterizing or describing objects by their mappings to other objects, rather than by their internal structure or construction. The modifier *naive* indicates that we do *not* aggressively axiomatize, and do *not* necessarily worry about reconciling categorical ideas with *set theory*.

Naive set theory is the typical background or foundation for constructing more complicated structures, in a century-old tradition. The modifier *naive* denotes an informal and non-axiomatic treatment, in contrast to versions of set theory specifically aimed at preventing paradoxes by limiting constructions. Part of the set theory tradition demands that *any* mathematical discussion be compatible with the explicit or implicit prohibitions of set theory.

Linear algebra in the broadest sense takes vector spaces over fields as prototypes for modules over general rings. *Decomposition* into eigenvectors or more complicated atomic pieces appears everywhere. Collections of modules over rings are the prototypes for *abelian categories*, including important examples that are *not* literal categories of modules, such as categories of *sheaves* or *complexes* of sheaves.

Functors arise immediately in category theory, being maps from one category to another, converting one type of object to another, and converting corresponding maps-of-objects from one type to the other. Homology

[1] The occasional definition of new objects as *symbols behaving a certain way* is a pale shadow of the idea of *characterization* by interactions, but made clumsy and opaque by entangling half-hearted constructions with under-specification.

and homotopy groups of topological spaces are dramatic first examples. Once we realize that most systematic procedures describe *functors*, we see that functors are *everywhere*.

Adjoint functors and adjunction relations: many important functors occur in *adjoint pairs*, satisfying an *adjunction relation*, in a sense a sort of inversion, but significantly different. Many *forgetful* or *lossy* functors, while boring themselves, have interesting adjoints. Yoneda's small lemma yields left exactness of right adjoints, and right exactness of left adjoints, once and for all.

Natural transformations are *maps of functors*, and allow formulation of the notion of *equivalence* of functors, the best version of *isomorphism of functors*.

Homology and cohomology arose as counting holes in geometric objects. Geometric interpretations are still important, but the underlying algebra is *universally* useful.

Extensions of rings appear in algebraic number theory and in algebraic geometry, with arithmetic or geometric motivations. Many important features are best described in categorical and homological terms.

Complexes eventually appear as more appropriate fundamental objects than *individual* modules. The structure theorem for finitely-generated modules over principal ideal domains is a prototype. *Projective* and *injective* resolutions are the fundamental *acyclic resolutions*.

These and other themes are profitably illustrated with examples, as follow.

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2. *Example: what is an indeterminate?*

What is an indeterminate x? We are not asking about a merely *unknown* literal number, as in the earliest use of algebra, but about the x that appears in polynomials and other functions in calculus.

The x is sometimes alleged to be a *variable* number, or an *arbitrary* number. It's not clear what *variable* means, but the vague idea of *motion* is generally compatible with the motivating issues of calculus. These are not bad pseudo-definitions to give calculus students.

However, the idea that x is merely a *variable number* does not match our use of it. For example, in the *Cayley-Hamilton theorem*[2] the indeterminate x is replaced by a *linear transformation*.

This suggests that x is something that can be *substituted for*. Elementary mathematics has no way to say this with any precision, but a categorical viewpoint succeeds, as follows.

Let R be a commutative ring, probably[3] with 1. An R-**algebra** A is a ring A that is also an R-module, and so that the R-module structure is compatible with the multiplication in A:

$$(r \cdot a)b \;=\; r \cdot (ab) \qquad\qquad (\text{for } r \in R,\ a, b \in A)$$

For example, R actually sitting inside the *center* of A gives A an R-algebra structure. But we do also want to allow some collapsing of R in the way that it acts on A. For example, $\mathbb{Z}/n$ is a $\mathbb{Z}$-algebra in the obvious fashion.

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[2] Recall the assertion of the Cayley-Hamilton theorem: let T be an endomorphism of a finite-dimensional vector space V, let $\chi_T(x)$ be the characteristic polynomial $\det(x \cdot I - T)$, with I the identity map on V. Then $\chi_T(T) = 0$. The non-proof of this by (falsely) claiming that $\chi_T(T) = \det(T - T) = \det 0 = 0$ is of interest, if only to provoke us to see why it's not a proof.

[3] Rings with 1 are more intuitive, and we *hope* that rings have 1, but occasionally there are good reasons to consider rings *without* units. But at the moment there's insufficient motivation to get embroiled with issues about lack of units.

An R-algebra *homomorphism* or *map* $f : A \to B$ is a ring homomorphism $f : A \to B$ that respects the R-algebra structure, in the reasonable sense that

$$f(r \cdot a) = r \cdot f(a) \qquad \text{(for } r \in R \text{ and } a \in A)$$

Given the idea of R-*algebra*, we can say what x is, albeit indirectly. A **free** R-algebra on x is an R-algebra F (thinking that $F = R[x]$, but don't want to prejudice ourselves by notation!) with distinguished element $x \in F$, having the **universal property** that, for *any* R-algebra A and choice of element $a_o \in A$, there *exists* a *unique* R-algebra map $f : F \to A$ such that $fx = a_o$.

This universal property *characterizes* the free algebra $R[x]$ *not* by telling what x *is*, but by telling how it *interacts* with other R-algebras.

The main idea is that we have *substituted* a_o for x, and that the rest of F accompanies x suitably. We do not exclude the possibility that something happens to the ring R, as in quotient maps $\mathbb{Z} \to \mathbb{Z}/n$.

Grant for a moment that the more elementary notion of *polynomial ring* $R[x]$ really *is* a free R-algebra on the generator x. Then for a field k we can certainly map x to an endomorphism T of a finite-dimensional k-vectorspace V, mapping the free k-algebra $k[x]$ to the k-algebra $k[T]$ of k-linear endomorphisms of V generated[4] by T. The more mundane case of replacing x by an element r_o of the ring R gives an R-algebra homomorphism $R[x] \to R$ mapping $x \to r_o$, called an **evaluation** homomorphism. A slightly subtler example is $\mathbb{Q}[x] \to \mathbb{Q}(\sqrt{2})$ by mapping $x \to \sqrt{2}$.

We might worry that the mapping characterization of the free algebra $F = R[x]$ accidently **under-specified** it. After all, we've not *directly* said anything about the internal structure of it. However, mildly amazingly, this seemingly vague and indirect characterization completely determines $F = R[x]$. Further, the style of the proof uses almost nothing about the specifics of the situation.

That is, suppose we had *two* free R-algebras on single generators: suppose A a free R-algebra on a_o, and B a free R-algebra on b_o. Then we claim that there is a *unique isomorphism* $j : A \to B$ of R-algebras such that $ja_o = b_o$.

Indeed, using the characterization of A, a_o as free R-algebra, there is a unique $\alpha : A \to B$ with $\alpha a_o = b_o$. On the other hand, using the characterization of B, b_o as free R-algebra, there is a unique $\beta : B \to A$ with $\beta b_o = a_o$. Then $\alpha \circ \beta$ is an R-algebra map of B to itself sending b_o to itself, and $\beta \circ \alpha$ is an R-algebra map of A to itself sending a_o to itself. Obviously we imagine that both these composites are the respective identity maps, which would prove the claim. But one small point is missing, as follows.

Universal objects have no proper automorphisms: we claim that the only R-algebra map of A to itself sending a_o to itself is the identity map on A. This and the corresponding fact for B and b_o would prove that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the respective identities, and, thus, that A, a_o and B, b_o are isomorphic. Using the mapping characterization of A and a_o in a seemingly silly way, there is a unique R-algebra map $A \to A$ sending a_o to a_o. On the other hand, the identity map on A certainly has this property. The *uniqueness* part of the characterization then says that there is no *other* such map than the identity. This proves the isomorphism of the two free R-algebras, and, in fact, proves that there is *only one* isomorphism of the two.

Notice that we did not use any properties of rings, of polynomials, or of anything else in proving uniqueness.

Neither characterization by mapping properties nor the uniqueness argument proves *existence* of a free R-algebra on x. Often existence of an object is proven by providing a *construction.* One benefit of knowing *uniqueness* is that it assures us that any construction will inevitably yield the same thing, up to unique isomorphism. That is, we would not have to *compare* two different constructions, because we know in advance that the outcomes will be isomorphic.

[4] As common in defining the phrase *generated by* inside a larger object, the k-algebra $k[T]$ of k-linear endomorphisms of V *generated by* T is the *intersection* of all k-subalgebras of $\mathrm{End}_k(V)$ containing T.

We'll not worry about a construction at this moment.

3. *Foundational dangers: set theory and universes*

Mapping-property definitions quantify over *all* things of a given sort. The collection of *all* things of a given sort is huge!

Even if we try to dodge that size issue by stipulating that we somehow only use one representative from each isomorphism class, the class of isomorphism classes is still huge. Maybe so large that certain people would say we're not *allowed* to call it a *class*, because its elements are classes already.

In fact, it is easy to misunderstand how large things can be, without a bit of attention to *set theory*. Paradoxical phenomena easily but perversely arise, such as the notorious

$$S \;=\; \{\text{sets}\, x \;:\; x \notin x\}$$

which can neither contain itself nor fail to contain itself. The problem can be viewed as proof that S *does not exist*, even though we appeared to define or describe it.

This is ominous, since a popular traditional attitude in mathematics is that if we can *describe* something, it must necessarily *exist*. Of course this is not correct, as in the case of an integer n such that $n < 4$ and $n > 4$. But thinking about *characterizing* rather than *constructing* does raise the ugly possibility that characterizations accidentally don't refer to anything.

This sort of worry is what motivates the part of set theory concerned with cautious *construction* under weak hypotheses, imagining that a successful construction thereby proves *existence* with the least extravagant assumptions possible.

Our discussion will take place in *naive* set theory, meaning without a formal list of *axioms*. The standard Zermelo-Fraenkel and vonNeumann-Bernays-Gödel axiom systems are given in an appendix.

An austere world of sets would begin with only the empty set $\phi = \{\}$ and allow or acknowledge only sets whose existence was guaranteed by reasonable constructions, or by axioms. Imagining that a finite list of things (sets) can be bundled up into a set, we can produce an endless list of sets, which von Neumann notated by the usual symbols for non-negative integers:

$$\begin{array}{ccccc} 0 & = & \phi & = & \{\} \\ 1 & = & \{0\} & = & \{\{\}\} \\ 2 & = & \{0,1\} & = & \{\{\},\{\{\}\}\} \\ 3 & = & \{0,1,2\} & = & \{\{\},\{\{\}\},\{\{\},\{\{\}\}\}\} \\ & & \ldots & & \end{array}$$

To collect all these into a single set apparently requires a different operation and justification, but we *want* to do so, and we have the **first infinite ordinal**

$$\omega_o \;=\; \{0,1,2,\ldots\} \;=\; 0 \cup 1 \cup 2 \cup \ldots \qquad \text{(an ascending union)}$$

The sense in which these processes can be continued *indefinitely* depends on (metaphysical/axiomatic) assumptions.

Thinking more positively, we have (at least) two processes: given an ordinal ω, form its **successor**, denoted $\omega + 1$,

$$(\text{successor of } \omega) \;=\; \omega \cup \{\omega\}$$

For example, we have $\omega + 2$, ..., $\omega + n$, ... As in the case of ω_o, we also take **limits**, *ascending* unions of ordinals

$$\lim_{j\in J} \alpha_j \;=\; \bigcup_{j\in J} \alpha_j \qquad \text{(with ordered set } J\text{, and } i < j \text{ implies } \alpha_i < \alpha_j)$$

An ordinal that is not a successor is a **limit ordinal**.

Not necessarily thinking only of *ordinals*, two sets x, y have the same *cardinality* if there is an *injection* $x \to y$ and there is an injection $y \to x$. In that situation, the Cantor-Schroeder-Bernstein theorem produces a *bijection* between the two, *without* the Axiom of Choice. Write $|x| = |y|$ for this. If x admits an injection to y, but y has no injection to x, write $|x| < |y|$.

There is also the **power set** PS of a given set S, that is, the set of all subsets. The Cantor diagonalization argument proves that $|PS| > |S|$.

The **cardinals** are representatives for *sizes*, and one style of choice of representatives, granting ourselves the Axiom of Choice, is from among ordinals, taking the *first* ordinal of a given size. From this viewpoint, the first infinite cardinal is the first infinite ordinal: $\aleph_0 = \omega_o$. Generally, whatever the representatives or model for them, cardinals can be indexed by ordinals. Thus, $\aleph_1$ is the first uncountable cardinal. We take the first uncountable ordinal ω_1 as a model for $\aleph_1$. Certainly $|P\aleph_0| \geq \aleph_1$, but the question of *equality* here is the **continuum hypothesis**, proven independent of ZFC by Paul Cohen, after Kurt Gödel showed consistency. Similarly, in general, for any ordinal ω,

$$|P\aleph_\omega| \;\geq\; \aleph_{\omega+1}$$

and equality would be part of a *generalized* continuum hypothesis.

A limit ordinal α is **regular** if it is *not* a limit over an index set of smaller cardinality.

An ordinal α is **inaccessible** if it is *regular* and $|\beta| < |\alpha|$ implies $|P\beta| < |\alpha|$ for ordinals β.

Existence of inaccessible ordinals does not follow from usual axioms of set theory.

In a world of sets where, *for some reason*, cardinalities of operations are constrained to be strictly less than the cardinality of an inaccessible ordinal α, operations with elements of α produce only other elements of α. That is, α is a **small model** for set theory, in the sense that it is a *set* (so is *small*), and the minimal operations of set theory can be performed in α, without taking us outside α.

Let α be an inaccessible ordinal. A world-of-sets in which every set has cardinality strictly less than that of α, but collections of objects (that is, **categories**) can have the cardinality of α, is roughly a **universe**.

The fact that α *is* a set in some *larger* world-of-sets is meant to be reassuring about the self-consistency and reasonableness of operations involving cardinalities as large as α, while pretending that such large operations are dangerous *in the model* exactly because they involve cardinalities as large as α, which is larger than any cardinal *in* the model.

Grothendieck would want to assume that **every set is contained in a universe**. This would make much of modern mathematics work more smoothly, but seems unapproachable with what we know now about large sets.

Unsurprisingly, worry about hierarchies of larger-and-larger inaccessible ordinals for category theory leads to artificialities and clumsinesses similar to those in Russell-Whitehead's *Principia*.

Some do claim that these considerations are necessary.

Some deny the necessity of comparing everything to set theory.

4. *Resolutions, complexes, snake lemma*

Linear algebra in the broad sense is modeled on vector spaces over fields, which behave very well.

For a ring R, a sequence of R-module maps

$$A \longrightarrow B \longrightarrow C$$

is *exact* when $\ker(B \to C) = \mathrm{Im}\,(A \to B)$. For example, for modules A, B, C over R, a chain of R-module maps

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a **short exact sequence** when each of the three **joints**

$$0 \longrightarrow A \longrightarrow B \qquad A \longrightarrow B \longrightarrow C \qquad B \longrightarrow C \longrightarrow 0$$

is exact, that is, when

$$\begin{array}{l} \ker(A \to B) \;=\; \mathrm{Im}\,(0 \to A) \;=\; \{0\} \\ \ker(B \to C) \;=\; \mathrm{Im}\,(A \to B) \\ \{0\} \;=\; \ker(C \to 0) \;=\; \mathrm{Im}\,(B \to C) \end{array}$$

In particular, in such a short exact sequence $A \to B$ is *injective* and $B \to C$ is *surjective*.

For a field k, every k-module is *free* on some set of generators. Indeed, more can be said. Every short exact sequence of k-vectorpaces

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

splits, in the sense that there is a map $s : C \to B$ giving a *left* inverse to $B \to C$, and such that

$$B \;=\; A \oplus sC$$

Symmetrically, there is a *right* inverse $p : B \to A$ such that

$$B \;=\; \mathrm{Im}\,(A) \oplus \ker p$$

All these things follow from discussion of *bases* for vector spaces: any basis for a subspace can be extended to a basis of the whole. Further, if one understands *ordinals* well enough to do *transfinite induction*, infinite-dimensional vector spaces can be dispatched in the same fashion. The argument is the *Lagrange replacement principle*.

As an example of a somewhat less trivial ring than a field we take $\mathbb{Z}$. Most $\mathbb{Z}$-modules are *not* free, but *finitely-generated* $\mathbb{Z}$-modules have an accessible structure theorem. One form of the structure theorem for finitely-generated $\mathbb{Z}$-modules M is that there is a unique list of non-negative integers $d_1, \ldots, d_n$ with the divisibility property $d_i | d_{i+1}$ such that

$$M \;\approx\; \mathbb{Z}/d_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/d_n\mathbb{Z}$$

The d_i are the *elementary divisors* of M. Part of this is captured in the following restatement: there exist finitely-generated *free* $\mathbb{Z}$-modules F_0 and F_1 such that the following sequence is *exact*:

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0 \qquad\qquad \text{(exact)}$$

Specifically, given an isomorphism as above, let m_i be the image in M of $1 \bmod d_i\mathbb{Z}$. Let F_0 be the *free* $\mathbb{Z}$-module on a set $\{f_1, \ldots, f_n\}$, and map $F_0 \to M$ *surjectively* by $f_i \to m_i$. The kernel of this map is

$$\mathbb{Z}d_1 f_1 \oplus \ldots \oplus \mathbb{Z}d_n f_n$$

Let F_1 be the free $\mathbb{Z}$-module on $g_1, \ldots, g_n$, and map $F_1 \to F_0$ by $g_i \to d_i f_i$. Thus, F_0 *injects* to F_1, and the image of F_1 in F_0 is exactly the kernel of $F_0 \to M$.

The exact sequence $0 \to F_1 \to F_0 \to M \to 0$ is called a **free resolution** of M.

Typically, as in the free resolution of $\mathbb{Z}/2$ given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \longrightarrow 0$$

there is no splitting or direct sum decomposition: the multiplication-by-2 map of $\mathbb{Z}$ to itself does *not* have a right inverse, and the image is *not* a direct summand. Similarly, there is no non-trivial map $\mathbb{Z}/ \longrightarrow \mathbb{Z}$. Thus, such a free resolution is *non-trivial* by comparison to what happens for vector spaces.

Polynomial rings $k[x_1, \ldots, x_n]$ over fields k are eminently reasonable rings, but for $n > 1$ are more complicated than $\mathbb{Z}$. A polynomial ring in a single variable is still a principal ideal domain, but larger polynomial rings are *not*, despite being unique factorization domains, by Gauss' lemma. For example, the maximal ideal generated by all the indeterminates $x_1, \ldots, x_n$ needs at least n generators.

Hilbert's syzygy theorem asserts that any finitely-generated $k[x_1, \ldots, x_n]$-module M has a free resolution

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of length at most n. That is, this sequence of maps is *exact*, each F_i is *free* over $k[x_1, \ldots, x_n]$, and the length of the resolution is n.

Thus, the **homological dimension** of $\mathbb{Z}$ is 1, while the homological dimension of $k[x_1, \ldots, x_n]$ is n.

By contrast, for the ring $R = \mathbb{Z}/4$, the R-module $\mathbb{Z}/2$ has *no* finite free resolution. The best we can do is the *infinite* free resolution

$$\cdots \xrightarrow{\times 2} \mathbb{Z}/4 \xrightarrow{\times 2} \mathbb{Z}/4 \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \longrightarrow 0$$

Thus, the *literal* size of $\mathbb{Z}/2$ gives no hint that it has infinite homological dimension over $\mathbb{Z}/4$.

[4.0.1] Remark: The contemporary idea is that we should think of an object *along with a resolution*, rather than an object in isolation.

The effect of simple operations on finitely-generated $\mathbb{Z}$-modules can be examined through their free resolutions. We consider *finitely-generated* $\mathbb{Z}$-modules because we know a good structure theorem for them. For a finitely-generated $\mathbb{Z}$-module M, and for a positive integer n, consider two related $\mathbb{Z}$-modules,

$$M^{[n]} \;=\; (\text{largest submodule on which } n \text{ acts by } 0) \;=\; \{m \in M \;:\; n \cdot m = 0\}$$

and

$$M_{[n]} \;=\; (\text{largest quotient on which } n \text{ acts by } 0) \;=\; M/nM$$

A related object is

$$M^{\text{tors}} \;=\; \text{torsion submodule of } M \;=\; \{m \in M \;:\; n \cdot m = 0 \text{ for some } n \neq 0\}$$

Note that $M \to M_{[n]}$ is a homomorphism, while $M \to M^{[n]}$ is not, in general. But the associations of $M_{[n]}$ and $M^{[n]}$ to M *are* examples of **functors** from $\mathbb{Z}$-modules to $\mathbb{Z}$-modules, because in addition to producing new *objects* (modules), they produce new *homomorphisms*: given a $\mathbb{Z}$-module homomorphism $f : M \longrightarrow N$, there are natural homomorphisms

$$f^{[n]} \;:\; M^{[n]} \longrightarrow N^{[n]} \qquad \text{by} \qquad f^{[n]}(m) \;=\; m$$

and

$$f_{[n]} \;:\; M_{[n]} \longrightarrow N_{[n]} \qquad \text{by} \qquad f_{[n]}(m + nM) \;=\; f(m) + nN$$

But, interestingly, *exactness* is *not* generally preserved by these functors. For example, from a free resolution of $\mathbb{Z}/d$,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times d} \mathbb{Z} \xrightarrow{\text{mod } d} \mathbb{Z}/d \longrightarrow 0 \qquad \text{(exact)}$$

Applying the $M \longrightarrow M_{[n]}$ functor, a not-necessarily-exact sequence arises:

$$0 \longrightarrow \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{(\times d)_{[n]}} \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{(\text{ mod } d)_{[n]}} \frac{\mathbb{Z}/d}{n\cdot\mathbb{Z}/d} \longrightarrow 0 \qquad \text{(possibly } \textit{not} \text{ exact)}$$

For example, with $n = d$, the multiplication-by-d from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/n\mathbb{Z}$ becomes the 0-map, which is certainly *not* injective. On the other hand, for n relatively prime to d, the injectivity is preserved. Bit all the *other* parts of the diagram *do* remain exact. That is, only *injectivity* at the left end is possibly lost. That is, we have a slightly smaller exact sequence

$$\frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{(\times d)_{[n]}} \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{(\text{ mod } d)_{[n]}} \frac{\mathbb{Z}/d}{n\cdot\mathbb{Z}/d} \longrightarrow 0 \qquad \text{(exact)}$$

Losing exactness in at most the left end of a short exact sequence is **right exactness** of the functor $M \longrightarrow M_{[n]}$. Similarly, the functor $M \to M^{[n]}$ loses exactness in at most the right end of an exact sequence, and is said to be **left exact**.

A natural question is what object X (depending on d and n) and what map $f : X \to \mathbb{Z}/n$ would *recover* exactness, by accommodating the non-injectiveness, giving exact

$$0 \longrightarrow X \xrightarrow{f} \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{(\times d)_{[n]}} \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{(\text{ mod } d)_{[n]}} \frac{\mathbb{Z}/d}{n\cdot\mathbb{Z}/d} \longrightarrow 0 \qquad \text{(exact?)}$$

In this simple case, there is a reasonable sort of symmetry, namely

$$0 \longrightarrow (\mathbb{Z}/d)^{[n]} \longrightarrow \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{(\times d)_{[n]}} \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{(\text{ mod } d)_{[n]}} \frac{\mathbb{Z}/d}{n\cdot\mathbb{Z}/d} \longrightarrow 0 \qquad \text{(exact)}$$

Indeed, the indicated quotients and kernels are readily computed and the exactness can be verified directly.

In fact, for *arbitrary* $\mathbb{Z}$-modules A, B, C in a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

it *turns out* that there is an exact sequence

$$0 \longrightarrow A^{[n]} \longrightarrow B^{[n]} \longrightarrow C^{[n]} \longrightarrow A_{[n]} \longrightarrow B_{[n]} \longrightarrow C_{[n]} \qquad 0 \qquad \text{(exact)}$$

Starting from that general result, for A and B *free* we have $A_{[n]} = 0$ and $B_{[n]} = 0$, degenerating into the simpler case.

This pattern is an instance of the **Snake Lemma**: suppose we are given a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{q} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{q'} & C' & \longrightarrow & 0 \end{array} \qquad \text{(with exact rows)}$$

which **commutes** in the sense that outcomes do not depend upon the route one traverses. That is, for example, starting with $a \in A$, applying i, then applying g produces the same element of B' as going the other way around that square, namely, first applying f, then i'. The general assertion is that

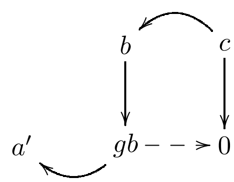
$$0 \longrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \frac{A'}{\mathrm{Im}\, f} \longrightarrow \frac{B'}{\mathrm{Im}\, g} \longrightarrow \frac{C'}{\mathrm{Im}\, h} \longrightarrow 0$$

All the maps are what one woulds expect, except possibly for the **connecting homomorphism** δ, which is constructed as follows.

Take $c \in C$ such that $hc = 0$. Since $B \to C$ is surjective, there is $b \in B$ mapping to c. Then $gb \in B'$ maps to 0 in C', by the commutativity of that square. Thus, since the kernel of $B' \to C'$ is the image of $A' \to B'$, there is $a' \in A'$ mapping to gb. Since A' injects to B', there is no ambiguity in choice of a' given gb. We declare

$$\delta c \;=\; a'$$

A little more pictorially,



The ambiguity of choice of b is exactly by the image iA of A, so qb is ambiguous exactly by $aiA = i'fA$. Thus, a' is ambiguous exactly by the image fA of A. This *defines* δ.

The following two little arguments exemplify **diagram chasing**: the conclusion is important, and the arguments are so broadly applicable that they might seem superficial. Further, when couched in most austere terms, suppressing as many details as possible, there are so few ways to go *wrong* that the *correct* line of argument is inescapable.

For *exactness* of $\ker g \longrightarrow \ker h \xrightarrow{\delta} A'/fA$, first take $b \in \ker g$. Then $\delta(qb) = 0$. That is, the composite of the two maps is 0. On the other hand, suppose $\delta c = 0 \in A'/fA$ for $c \in \ker h$. That is, with $b \in B$ mapping to c, $gb = i'a'$ with $a' = fa$. It is *not* necessarily true that $ia = b$, but $g(ia) = gb$ because the left square commutes. That is, $g(b - ia) = 0$. Also, still $q(b - ia) = c$, because the top row is exact. Thus, c is in the image of $\ker g$, proving exactness at that joint.

For *exactness* of $\ker h \xrightarrow{\delta} A'/fA \longrightarrow B'/gB$, first take $c \in C$. As in the construction of δ, there is $b \in B$ such that $qb = c$, and then $a' \in A'$ such that $i'a' = gb$, and $\delta c = a' + fA$. Since $i'a' \in gB$, the map to B'/gB sends δc to 0. This proves that the composition of the two maps is 0. On the other hand, given $a' \in A'$ mapping to 0 in B'/gB, there is $b \in B$ such that $gb = i'a'$. Since the right-hand square *commutes*, $h(qb) = q'(gb) = 0$, since the bottom row is exact. That is, the element $c = qb$ is in $\ker h$, and, by this discussion, $\delta c = a'$. This gives the desired exactness.

It would be tiresome to check directly the exactness at all other joints of this diagram. In fact, left exactness of $M^{[n]}$ and right exactness of $M_{[n]}$ are guaranteed by the **adjointness** relation

$$\mathrm{Hom}_{\mathbb{Z}}(M_{[n]}, N) \;\approx\; \mathrm{Hom}_{\mathbb{Z}}(M, N^{[n]}) \qquad\qquad \text{(for } M, N\ \mathbb{Z}\text{-modules)}$$

via **Yoneda's lemma**, discussed a little later.

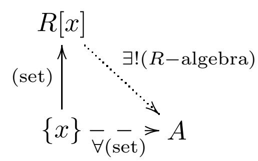
In fact, *all the above works for very general reasons*, so applies to modules over *any* commutative ring R, and finite-generation is not necessary.

---

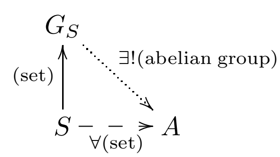
5. *Pictorial interlude, tensors, etc.*

Pictorial/diagrammatical presentations of universal mapping properties are decisive mnemonics, *and* illustrate that the *shapes* of the diagrams coming from very different situations nevertheless strongly resemble each other.

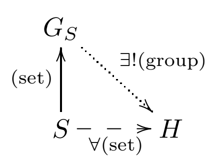
[5.1] Indeterminates, free algebras The polynomial ring $R[x]$ on x over a commutative ring R is a **free** R **algebra** on x, so includes an obvious *set* map $\{x\} \to R[x]$, and any set map $\{x\} \longrightarrow A$ to a commutative R-algebra A produces a unique compatible R-algebra map $R[x] \longrightarrow A$. The diagram that conveys this is



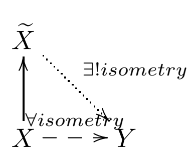
[5.2] Free abelian groups A free abelian group G_S on a set S includes a set map $S \to G_S$ such that any set map $S \longrightarrow A$ to an abelian group A produces a unique compatible abelian-group map $G_S \longrightarrow A$. The diagram that conveys this is



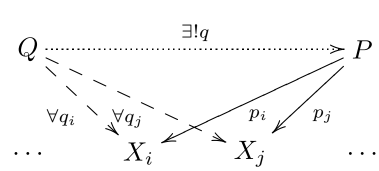
[5.3] Free not-necessarily-abelian groups A free (not-necessarily abelian) group G_S on a set S includes a set map $S \to G_S$ such that any set map $S \longrightarrow H$ to a group H produces a unique compatible group map $G_S \longrightarrow H$. The diagram that conveys this is



[5.4] Completions of metric spaces Given a metric space X, a **completion** of A is a metric space $\widetilde{X}$ and an isometric map $X \longrightarrow \widetilde{X}$ such that, for every isometric map $X \longrightarrow Y$ to a *complete* metric space Y, there is a unique isometric map $\widetilde{X} \longrightarrow Y$ through which $X \longrightarrow Y$ factors. That is, we have commutative diagrams



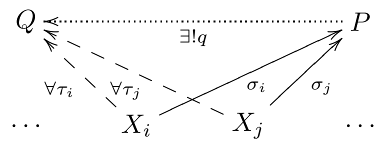
[5.5] Products The *product* of a collection $\{X_i : i \in I\}$ of objects X_i indexed by a set I is an object P and **projections** $p_i : P \to X_i$ such that, given another object Q and a family of maps $q_i : Q \to X_i$, there is a unique $q : Q \to P$ such that all the maps q_i **factor through** q, in the sense that $q_i = p_i \circ q$. That is, the following diagram commutes:



where it is understood that the single map $q : Q \longrightarrow P$ must meet all the conditions $q_i = p_i \circ q$ simultaneously.

Note that this definition of product makes sense for *any* specific type of object, with all the mappings being the corresponding type: sets and set maps, groups and group homomorphisms, topological spaces and continuous maps, etc.

[5.6] Coproducts The *coproduct* of a collection $\{X_i : i \in I\}$ of objects X_i indexed by a set I is an object C and maps $\sigma_i : X_i \to C$ such that, given another object D and a family of maps $\tau_i : X_i \to D$, there is a unique $\tau : C \to D$ such that all the maps τ_i **factor through** τ, in the sense that $\tau_i = \tau \circ \sigma_i q$. That is, the following diagram commutes:

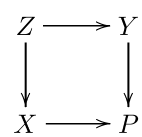


where it is understood that the single map $q : Q \longrightarrow P$ must meet all the conditions $q_i = p_i \circ q$ simultaneously.

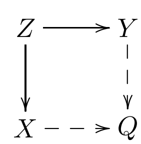
Again, this definition of product makes sense for *any* specific type of object, with all the mappings being the corresponding type.

[5.6.1] Remark: The definition of *coproduct* is that of *product* with arrow directions reversed.

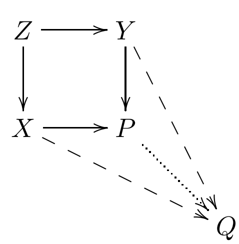
[5.7] Pushouts For two objects X, Y and given maps $Z \to X$ and $Z \to Y$, a **pushout** of X, Y along Z is an object P and maps $X \to P$ and $Y \to P$ such that the following diagram commutes



and such that, for every pair of maps $X \to Q$ and $Y \to Q$ with a commuting diagram



there is a unique $P \longrightarrow Q$ such that we have a commuting diagram

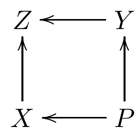


Pushouts can be defined for larger collections of objects X_i each with a map $Z \to X_i$.

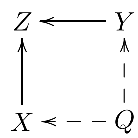
[5.7.1] Example: Pushouts of *topological spaces* are also called **glue-ings**.

[5.7.2] Example: Pushouts of *groups* are also called **amalgamated products**.

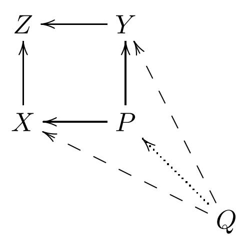
[5.8] Pullbacks Reversing all the arrows in the definiton of pushouts, we have *pullbacks:* For two objects X, Y and given maps $X \to Z$ and $Y \to Z$, a **pullback** of X, Y along Z is an object P and maps $P \to X$ and $P \to Y$ such that the following diagram commutes



and such that, for every pair of maps $Q \to X$ and $Q \to Y$ with a commuting diagram



there is a unique $Q \longrightarrow P$ such that we have a commuting diagram



Pullbacks can be defined for larger collections of objects X_i each with a map $X_i \to Z$.

[5.9] Tensor products of modules over commutative rings Let R be a *commutative* ring with 1. An R-**bilinear map** $B : M \times N \longrightarrow V$ with R-modules M, N, V is a map that is R-linear in each of its two arguments *separately.* That is,

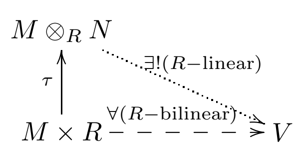
$$B(am + bm', n) \;=\; aB(m, n) + bB(m', n) \qquad \text{(with } a, b \in R,\ m, m' \in M,\ n \in N)$$

and

$$B(m, an + bn') \;=\; aB(m, n) + bB(m, n') \qquad \text{(with } a, b \in R,\ m \in M,\ n, n' \in N)$$

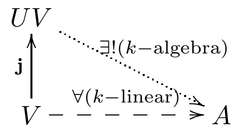
Inner products on vector spaces provide a common example of bilinear maps.

The **tensor product** $M \otimes_R N$ converts bilinear maps on $M \times N$ to linear maps: there is an R- *bilinear* $\tau : M \times N \longrightarrow M \otimes_R N$ such that the following diagram commutes:



The *image* of $m \times n$ in $M \otimes_R N$ is usually denoted $m \otimes n$.

[5.10] Universal/tensor algebras Let V be a vector space over a field k. The **universal algebra** (sometimes called *tensor algebra*) of V is an associative k-algebra UV and a k-linear map $j : V \longrightarrow UV$ such that, for any k-linear $V \longrightarrow A$ for a k-algebra A, there is a unique k-algebra map $UV \longrightarrow V$ making the following diagram commute:



The reason UV is often called the **tensor** algebra over V, and denoted $\bigotimes^\bullet V$, is that UV can be *constructed* from tensor powers of V, as follows.

First, grant *existence* of tensor products. Tensor products were *characterized* above.

Let $T = \bigoplus_{n\geq 0} \bigotimes^n V$, where $\bigotimes^0 = k$ and

$$\bigotimes^n V \;=\; \underbrace{V\otimes\ldots\otimes V}_{n} \qquad\qquad (\text{for } n\geq 1)$$

The addition is just the addition in the direct sum, and the multiplication is the bafflingly innocuous

$$(v_1\otimes\ldots\otimes v_m)\otimes(w_1\otimes\ldots\otimes v_n) \;=\; v_1\otimes\ldots\otimes v_m\otimes w_1\otimes\ldots\otimes v_n$$

Proof must be given that this construction succeeds.

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6. *Adjoint pairs of functors*

Everything is an adjoint.

What does this mean? Is it a good thing, or a bad thing?

[6.1] Indeterminates and free algebras The characterization of a polynomial ring $R[S]$ on a *set* S (of so-called *indeterminates*) as a **free** commutative **R-algebra** on S can be written as an isomorphism of sets of homomorphisms, an example of an **adjunction**:

$$\mathrm{Hom}_{R-\mathrm{algs}}(R[S],A) \;\approx_{\mathrm{sets}}\; \mathrm{Hom}_{\mathrm{sets}}(S,A) \qquad\qquad (\text{for sets } S,\ R\text{-algebras } A)$$

where the subscripts on *Hom* tell what *kind* of homomorphisms. Note that in $\mathrm{Hom}_{\mathrm{sets}}(S,A)$ the ring structure of A is **forgotten**. Only the **underlying set** is remembered when considering maps of $\{x\}$ to A. Since this *forgetting* has some significance, it should not always be suppressed from the notation: let F be the **forgetful functor**

$$F \;:\; \{R-\text{algebras}\} \;\longrightarrow\; \{\text{sets}\} \qquad\qquad (\text{taking underlying sets})$$

Then

$$\mathrm{Hom}_{R-\mathrm{algs}}(R[S],A) \;\approx_{\mathrm{sets}}\; \mathrm{Hom}_{\mathrm{sets}}(S,FA) \qquad\qquad (\text{for sets } S \text{ and } R\text{-algebras } A)$$

[6.2] Free modules Similarly, the **free R-module** M_X on generators X satisfies an adjunction

$$\mathrm{Hom}_{R-\mathrm{mods}}(M_X,M) \;\approx_{\mathrm{sets}}\; \mathrm{Hom}_{\mathrm{sets}}(X,FM) \qquad\qquad (\text{for } R\text{-modules } M)$$

where now F is the *forgetful functor* taking an R-module to its underlying set.

[6.3] Adjunctions, adjoints The above examples of **free objects** are straightforward, but do illustrate the adjunction pattern

$$\mathrm{Hom}_{\mathrm{objects}}(\text{free obj}(X),\text{object}) \;\approx_{\mathrm{sets}} \mathrm{Hom}_{\mathrm{sets}}(X,\ F(\text{object})) \qquad\qquad (\text{forgetful functor } F \text{ to sets})$$

That is, the formation of the free R-algebra $R[S]$ is **adjoint** to the forgetful functor from R-algebras to sets.

Other useful functors are adjoints to *less*-forgetful functors.

The following example clarifies notions such as *complexification* of real vector spaces.

[6.4] Extension of scalars, complexification Let K be a big field and k a subfield. Given a vector space V over K, certainly we can consider V as a vector space over k by simply *forgetting* that scalar multiplication by elements of K (other than elements of k) is possible. Thus, we have a forgetful functor

$$F \;:\; \{K - \text{vectorspaces}\} \longrightarrow \{k - \text{vectorspaces}\}$$

A **left adjoint** L to F would be a functor

$$L \;:\; \{k - \text{vectorspaces}\} \longrightarrow \{K - \text{vectorspaces}\}$$

such that

$$\mathrm{Hom}_{K-\text{v.s.s}}(LU, V) \approx \mathrm{Hom}_{k-\text{v.s.s}}(U, FV) \qquad (\text{for } k\text{-vectorspaces } U,\ K\text{-vectorspaces } V)$$

A **right adjoint** R to F would be a functor

$$R \;:\; \{k - \text{vectorspaces}\} \longrightarrow \{K - \text{vectorspaces}\}$$

such that

$$\mathrm{Hom}_{k-\text{v.s.s}}(FV, U) \approx \mathrm{Hom}_{K-\text{v.s.s}}(V, RU) \qquad (\text{for } k\text{-vectorspaces } U,\ K\text{-vectorspaces } V)$$

To say that these R, L are *functors* does entail the requirement that they act not only on *objects* but on *homomorphisms*, converting

$$f \;:\; U \longrightarrow U' \qquad (\text{homomorphism of } k\text{-vectorspaces})$$

into

$$Rf \;:\; RU \longrightarrow RU' \qquad (\text{homomorphism of } K\text{-vectorspaces})$$

and

$$Lf \;:\; LU \longrightarrow LU' \qquad (\text{homomorphism of } K\text{-vectorspaces})$$

and respecting composition.

When $k = \mathbb{R}$ and $K = \mathbb{C}$, these adjoints are **complexifications.** [5]

These clear characterizations of the adjoints do not prove *existence*. In fact, adjoint functors can be constructed from already-familiar [6] objects:

$$RU = \mathrm{Hom}_k(K, U) \qquad LU = K \otimes_k U \qquad (\text{for } k\text{-vectorspaces } U)$$

[5] An opaque traditional definition of *complexification* declares that we are somehow allowed to take linear combinations with *complex* coefficients, rather than merely real. It's not clear what this means, and it's not clear how to correctly manipulate the resulting objects. In fact, the present mildly categorical characterization captures the intent, which had been difficult to convey with a smaller vocabulary.

[6] In principle, tensor products are familiar, but often an introductory version is inadequate for real use. The universal mapping property above is a correct indication, but one must consider some of the ramifications.

That is, we have *adjunctions*

$$\mathrm{Hom}_K(K \otimes_k U, V) \approx \mathrm{Hom}_k(U, FV) \qquad \mathrm{Hom}_k(FV, U) \approx \mathrm{Hom}_K(V, \mathrm{Hom}_k(K, U))$$

In fact, this works the same way for a commutative ring k and commutative k-algebra K, both with units.

Further, when we allow non-commutative rings or groups to act on modules, adjoints to certain forgetful functors are **induced** modules, and the adjunction relation is **Frobenius reciprocity**. We'll come back to these examples later.

[6.5] Annihilated and co-annihilated modules Let R be a commutative ring with unit. For an ideal I of R and a module M over R, let

$$\begin{array}{ccccc} M^I & = & \text{(largest submodule on which } I \text{ acts by 0)} & = & \{m \in M : i \cdot m = 0 \text{ for all } i \in I\} \\ M_I & = & \text{(largest quotient module on which } I \text{ acts by 0)} & = & M/I \cdot M \end{array}$$

Then there is the adjunction relation

$$\mathrm{Hom}_R(M_I, N) \approx \mathrm{Hom}_R(M, N^I)$$

[6.6] Smallest Yoneda lemma For functors such as $M \to M^I$ and $M \to M_I$ that are **additive** in the sense of *preserving direct sums*, an adjunction relation has an important corollary: a left adjoint L is *right exact*, meaning that from

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \qquad \text{(exact)}$$

we have a slightly smaller exact sequence

$$LA \longrightarrow LB \longrightarrow LC \longrightarrow 0 \qquad \text{(exact)}$$

Symmetrically, a right adjoint is *left exact*, meaning that from that short exact sequence we have a slightly smaller exact sequence

$$0 \longrightarrow RA \longrightarrow RB \longrightarrow RC \qquad \text{(exact)}$$

This universal partial-exactness not only eliminates the need for repetitive arguments, but, in fact, often provides proof where a direct computation is possible but ugly.

The main idea in the proof is a very small instantiation of **Yoneda's Lemma**, in the form: given $A \to B \to C$, if $\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$ is exact for every X, then $A \to B \to C$ is exact.

[6.6.1] Remark: Exactness of $A \to B \to C$ does *not* imply exactness of $\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$.

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7. *Counting holes: algebraic topology*

The linear algebra involved in counting holes of various dimensions in geometric objects was first recognized as such by Emmy Noether: she observed that **homology groups** of topological spaces were really *groups*. Prior to that, the usual practice was to attach a list of numerical invariants, effectively the elementary divisors of (abelian) homology groups. The numerical-invariant viewpoint obscured the functorial nature of homology.

The algebra arising from homology of topological spaces has taken its name from the homology theory of those spaces.

Basic homology theory of topological spaces has given us conceptually vivid proofs of many fundamental geometric results: the Jordan curve theorem, generalized to the analogous result for $(n-1)$-spheres injecting to Euclidean n-space, *invariance of domain* (that is, the non-homeomorphic-ness of Euclidean spaces of disparate dimensions), and others.

[iou...]

[7.1] Homology of a hollow tetrahedron

[iou...]

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8. *Appendix: set theory axioms*

In 1908 Ernst Zermelo attempted an axiomatic description of set theory. His ideas were revised a bit by Abraham Fraenkel and Thoralf Skolem in 1922, and then Zermelo returned to the issue in 1930. The **Zermelo-Frankel** axioms are as follows. These axioms attempt to be minimalist, asserting just enough *existence* hypotheses to make usual mathematics work.

First, *everything* is a set. There is a single primitive relation $x \in y$, *being an element of*, that two sets x, y may or may not satisfy.

Two sets are *disjoint* if they have *no* elements in common. That is, x and y are disjoint if $z \in x$ implies $z \notin y$.

Extensionality: two sets are *equal* if they have the same elements. From this, two sets are *elements of* the same sets if they have the same elements.

Regularity/Foundation: Every non-empty set S contains an element x such that x and S are *disjoint*. This disallows $x \in x$.

Restricted comprehension: Given a set S and a *property*[7] P, there *exists* a set

$$\{x \in S \;:\; x \text{ has property } P\}$$

Unsurprisingly, to avoid self-reference, the property P must not refer to S.

Pairing: Given two sets, there is a set $\{x, y\}$, that is, whose elements are *only* x, y. That is, $z \in \{x, y\}$ implies $z = x$ or $z = y$ (and the possibility that $x = y$ is not excluded).

Union: Given a set S of sets, the *union* of the sets in S is a set: there exists a set

$$\bigcup_{x \in S} x \qquad\qquad (\text{also sometimes denoted } \bigcup S)$$

That is, $y \in \bigcup S$ if and only if there exists $x \in S$ such that $y \in x$.

Define an **ordered pair** (x, y) to be $\{\{x\}, \{x, y\}\}$. One readily proves that $(x, y) = (x', y')$ if and only if $x = x'$ and $y = y'$. The **cartesian product** $A \times B$ is

A **function** $f : A \to B$ from one set to another is a set f such that $x \in f$ implies $x = (a, b)$ for some $a \in A$ and $b \in B$, and such that for *every* $a \in A$ there exists a *unique* $x \in f$ with $x = (a, b)$. Of course, write $f(a) = b$.

Collection/replacement schema: For a function $f : A \to B$, the *range* $\{f(a) : a \in A\}$ is a subset of B.

[7] The notion of *property* can be formalized to refer to grammatically correct predicates in *first-order predicate logic*.

Infinity: There exists a set S containing the empty set and such that for $x \in S$ also $\{x\} \in S$.

The latter axiom incidentally has stipulated the existence of the **empty set**.

A set A is a **subset** of a set B if $x \in A$ implies $x \in B$, and write this $A \subset B$.

Power set: For a given set S, there exists a set PS with the property that $T \subset S$ if and only if $T \in PS$.

These are the Zermelo-Fraenkel axioms, and the set theory using this as a basis is **ZF set theory**.

For many purposes, a further axiom is added, giving us **ZFC set theory**:

Axiom of Choice: Given a set S of sets, with any two distinct x, y in S *disjoint*, there is a set C containing exactly one element from each $x \in S$.

The Axiom of Choice has several useful logical equivalents: Well-ordering Axiom, Zorn's Lemma, Hausdorff Maximality Principle.

In the 1920s John von Neumann developed a sort of set theory with *functions*, not sets, as primitives. From the 1930s through the 1950s, Paul Bernays rewrote it as a set theory, and Kurt Gödel contributed to it around 1940. The operational point is that this set theory allows **classes**, meaning collections that are too large to be *sets*. For example, by design, the collection of *all sets* cannot be a set, because it violates the regularity/foundation axiom.

Proper classes: The collection of all sets is a class. A class C is a set if and only if there is *no* bijection between C and the class of all sets. In addition to essentially the same axioms for sets, there is **Class Extensionality:** two classes are equal if and only if they have the same elements. There is **Class Regularity/Foundation:** a non-empty class is disjoint from one of its elements. This is **NBG set theory**.

In the 1960s, modern algebraic geometry reformulated by Alexandre Grothendieck and his school appeared to demand *hierarchies* of larger and larger class-like objects. At roughly the same time, *large cardinals* had been studied enough to know that existence of inaccessibles could not be proven from ZFC. Grothendieck suggested a formalization sufficient for the needs of algebraic geometry, as follows. A **universe** is a set U closed under natural set-theoretic operations:

$\phi \in U$
If $u \in U$, then $u \subset U$
If $u \in U$, then $\{u\} \in U$
If $u \in U$, then the power set of u is in U
If $I \in U$ and $u_i \in U$ for all $i \in I$, then $\bigcup_{i \in I} u_i \in U$
$\{0, 1, 2, \ldots\} \in U$

Grothendieck's axiom is **every set is contained in a universe**.
