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Adjoints, naturality, exactness, small Yoneda lemma

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The best way to understand or remember left-exactness or right-exactness of an additive [1] functor is to observe that it is a *right adjoint* or *left adjoint*. Many familiar functors occur in pairs whose adjointness is *obvious* once observed. MacLane and others have quipped *Everything's an adjoint.*

The proof that left adjoints are right-exact, and that right adjoints are left-exact, uses a small incarnation of *Yoneda's Lemma*, and illustrates the importance of *naturality* of isomorphisms.

Throughout, it suffices to think of categories of modules, although the arguments apply more generally.

- $\mathrm{Hom}(X, -)$ is left exact
- Adjoints and naturality
- Yoneda lemma
- Half-exactness of adjoints

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1. $\mathrm{Hom}(X, -)$ *is left exact*

Everything else will reduce to the straightforward left-exactness of $\mathrm{Hom}(X, -)$.

[1.0.1] Theorem: The functor $\mathrm{Hom}(X, -)$ is left exact. That is, a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}(X, A) \xrightarrow{i\circ -} \mathrm{Hom}(X, B) \xrightarrow{q\circ -} \mathrm{Hom}(X, C)$$

where the induced maps are by the obvious post-compositions with i and q. Similarly, the contravariant Hom functor $\mathrm{Hom}(-, X)$ gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}(C, X) \xrightarrow{-\circ q} \mathrm{Hom}(B, X) \xrightarrow{-\circ i} \mathrm{Hom}(A, X)$$

where the induced maps are the pre-compositions with i and q.

Proof: For $f \in \mathrm{Hom}(X, A)$, $i \circ f = 0$ implies $(i \circ f)(x) = 0$ for all $x \in X$, and then $f(x) = 0$ for all x since i is injective. Thus, $\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B)$ is injective, giving exactness at the left joint.

Since $q \circ i = 0$, any $f \in \mathrm{Hom}(X, A)$ is mapped to $0 \in \mathrm{Hom}(X, C)$ by $f \to q \circ i \circ f$. That is, the image of $i \circ -$ is contained in the kernel of $q \circ -$. On the other hand, when $g \in \mathrm{Hom}(X, B)$ is mapped to $q \circ g = 0$ in $\mathrm{Hom}(X, C)$, we have

$$g(X) \subset \ker q \;=\; \mathrm{Im}\, i$$

Since i is injective, it is an isomorphism to its image, so there is an inverse $i^{-1} : i(A) \to A$. Since $g(X) \subset \mathrm{Im}\, i$ we can define

$$f \;=\; i^{-1} \circ g \in \mathrm{Hom}(X, A)$$

[1] A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ of categories whose hom-sets are *abelian groups additive* when the map on morphisms $\mathrm{Hom}(A, B) \to \mathrm{Hom}(FA, FB)$ given by F is a homomorphism of abelian groups. This also entails $F(A \oplus B) \approx FA \oplus FB$. These isomorphisms are required to be *natural*.

Certainly $i \circ f = g$, so the kernel is contained in the image. This gives exactness at the middle joint, and the left exactness. The exactness of the other Hom is similar. ///

[1.0.2] Remark: The functor $\mathrm{Hom}(X, -)$ is *not* right exact. For example, with

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

with an integer $n > 1$, with $X = \mathbb{Z}/n$ there is no non-zero map of the torsion abelian group X to the free abelian group $\mathbb{Z}$. That is, the right joint in the following is not exact:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}) \arrow[r, "\times n"] \arrow[d, equal] & \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/n) \arrow[r, dashed] \arrow[d, equal] & 0 \\ & 0 & 0 & \mathbb{Z}/n & \end{tikzcd}$$

Similarly, $\mathrm{Hom}(-, X)$ is not right exact. [2]

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2. *Adjoints and naturality*

Two functors R and L are **mutually adjoint** when there is an **adjunction**

$$\mathrm{Hom}(LA, B) \approx \mathrm{Hom}(A, RB) \qquad \text{(for all } A, B)$$

The functor R is a **right adjoint**, and L is a **left adjoint**. The adjunction isomorphism is required to be *natural* or *functorial*, in the sense that, for each pair of morphisms $f : A' \to A$ and $g : B \to B'$ (yes, from A' to A, but from B to B') we have a commutative diagram [3]

$$\begin{tikzcd} \mathrm{Hom}(LA, B) \arrow[r, "\approx"] \arrow[d, "g \circ (-) \circ Lf"'] & \mathrm{Hom}(A, RB) \arrow[d, "Rg \circ (-) \circ f"'] \\ \mathrm{Hom}(LA', B') \arrow[r, "\approx"] & \mathrm{Hom}(A', RB') \end{tikzcd}$$

where the notation for pre-composition and post-composition is

$$g \circ (-) \circ Lf \;:\; F \longrightarrow g \circ F \circ Lf \qquad \text{(for } F \in \mathrm{Hom}(LA, B))$$

and

$$Rg \circ (-) \circ f \;:\; F \longrightarrow Rg \circ F \circ f \qquad \text{(for } F \in \mathrm{Hom}(A, RB))$$

In categories whose hom sets have additional structure, such as that of *abelian group*, the natural isomorphisms of hom sets are required to respect that additional structure, and satisfaction of this requirement is usually obvious.

We prove naturality in two examples of adjoint pairs.

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[2] Left-exactness and right-exactness have immediate sense for additive functors F between categories of modules. The notion of exactness of a sequence $A \to B \to C$ of two maps in a more general category makes sense when there are *kernels* and *co-kernels* (essentially, *quotients*) in the category. This is part of the notion of *abelian* category.

[3] Assembling these naturality isomorphisms into larger diagrams is critical in proving the left/right-exactness from adjointness.

[2.1] Annihilated and co-annihilated modules Let Λ be a ring with 1. Consider the category of Λ-modules and Λ-module homomorphisms. Fix an ideal I in Λ. A Λ-module N is **annihilated** by I if $i \cdot n = 0$ for all $i \in I$ and $n \in N$. For convenience, say that such N is I-**null**.

Given a Λ-module M, M^I is a Λ-module with a Λ-module map $j : M^I \to M$ through which every map $N \to M$ from an I-null Λ-module N factors. Dually, M_I is an Λ-module with Λ-module map $q : M \to M_I$ through which every map $M \to N$ to an I-null Λ-module N factors.

The *construction* (proof of existence) of M^I and M_I element-wise is easy, as sub-object and quotient, respectively:

$$\begin{cases} M^I &= \{m \in M \; : \; i \cdot m = 0 \text{ for all } i \in I\} \\ \\ M_I &= M/(I \cdot M) \end{cases}$$

[2.1.1] Proposition: The functors $LM = M_I$ and $RM = M^I$ are adjoints:

$$\mathrm{Hom}_\Lambda(LA, B) \;\approx\; \mathrm{Hom}_\Lambda(A, RB)$$

Proof: The isomorphism of hom-sets is the obvious $f \to f \circ q$, where $q : A \to A_I$ is the quotient map. The fact that $f \circ q$ has image inside the subobject $RB = B^I$ follows from the fact that $f : A_I \to B$ has image inside B^I, which follows from the fact that I acts by 0 on A_I.

The issue of interest is the naturality. Let $\alpha : A' \to A$ and $\beta : B \to B'$ be Λ-module homomorphisms. Let $q' : A' \to A'_I$ be the quotient map. That $A \to A_I$ is a *functor* implicitly claims that there is a homomorphism $\alpha_I : A'_I \to A_I$. Indeed, the composite

$$A' \longrightarrow A \longrightarrow A_I$$

must factor through A'_I, by its universal property, yielding a unique $\alpha_I : A'_I \longrightarrow A_I$ fitting into the commutative diagram

$$\begin{tikzcd} A' \arrow[r, "q'"] \arrow[d, "\alpha"'] & A'_I \arrow[d, "\alpha_I"] \\ A \arrow[r, "q"'] & A_I \end{tikzcd}$$

Similarly, there is $\beta^I : B^I \to {B'}^I$ fitting into a commutative diagram, namely, β^I is simply the restriction of β to B:

$$\begin{tikzcd} B^I \arrow[r, "i"] \arrow[d, "\beta^I"'] & B \arrow[d, "\beta"] \\ B'^I \arrow[r, "i'"'] & B' \end{tikzcd}$$

The *naturality* is the commutativity of

$$\begin{tikzcd} \mathrm{Hom}(A_I, B) \arrow[r, "\approx"] \arrow[d, "\beta\circ(-)\circ\alpha_I"'] & \mathrm{Hom}(A, B^I) \arrow[d, "\beta^I\circ(-)\circ\alpha"] \\ \mathrm{Hom}(A'_I, B') \arrow[r, "\approx"] & \mathrm{Hom}(A', B'^I) \end{tikzcd}$$

Note that at this point we do not give the top and bottom edge isomorphisms explicitly. This is clarified in the following.

The desired commutative diagram expands to a larger diagram upon making explicit the components of the isomorphism whose naturality is at issue. Namely, we claim that the following is commutative:

$$\begin{tikzcd}
\mathrm{Hom}(A_I, B) \arrow[d, "\beta\circ-"'] & \mathrm{Hom}(A_I, B^I) \arrow[l, "i\circ-"', "\approx"] \arrow[r, "-\circ q", "\approx"'] \arrow[d, "\beta^I\circ-"'] & \mathrm{Hom}(A, B^I) \arrow[d, "\beta^I\circ-"'] \\
\mathrm{Hom}(A_I, B') \arrow[d, "-\circ\alpha_I"'] & \mathrm{Hom}(A_I, B'^I) \arrow[l, "i\circ-"', "\approx"] \arrow[r, "-\circ q", "\approx"'] \arrow[d, "-\circ\alpha_I"'] & \mathrm{Hom}(A, B'^I) \arrow[d, "-\circ\alpha"'] \\
\mathrm{Hom}(A'_I, B') & \mathrm{Hom}(A'_I, B'^I) \arrow[l, "i'\circ-"', "\approx"] \arrow[r, "-\circ q'", "\approx"'] & \mathrm{Hom}(A', B'^I)
\end{tikzcd}$$

The three horizontal maps on the left half are the definition of $(-)^I$, while the three horizontal maps on the right half are the definition of $(-)_I$. Each vertical map is the image of a morphism by a *Hom* functor.

Commutativity of the upper left square follows from applying $\mathrm{Hom}(A_I, -)$ to the square expressing functoriality of $(-)_I$. Similarly, commutativity of the lower right square follows from applying $\mathrm{Hom}(-, B'^I)$ to the square expressing functoriality of $(-)_I$.

The upper right and lower left squares commute because composition of homs is associative.

Since the directions of the horizontal maps in each row are in opposite directions, it is only because they are *isomorphisms* that they can be composed. The composition along the top edge, and the composition along the bottom edge, are the isomorphisms in the assertion of adjunction. The adjunction square is obtained by keeping only the four outer corner *Hom*s, and the composite maps along the outer edges. ///

[2.2] $\mathrm{Hom}(X, -)$ and $(-) \otimes X$

This more complicated example is still basic.

[2.2.1] Proposition: For all $\mathbb{Z}$-modules A, X, B there is a *natural* isomorphism of $\mathbb{Z}$-modules

$$\mathrm{Hom}(A \otimes X, B) \approx \mathrm{Hom}(A, \mathrm{Hom}(X, B))$$

Proof: Once one knows that there is such an isomorphism, there is only one possibility that makes any sense: given $\Phi \in \mathrm{Hom}(A \otimes X, B)$, define $\varphi_\Phi \in \mathrm{Hom}(A, \mathrm{Hom}(X, B))$ by

$$\varphi_\Phi(a)(x) = \Phi(a \otimes x)$$

Conversely, given $\varphi \in \mathrm{Hom}(A, \mathrm{Hom}(X, B))$, define $\Phi_\varphi \in \mathrm{Hom}(A \otimes X, B)$ by

$$\Phi_\varphi(a \otimes x) = \varphi(a)(x)$$

and extending by linearity. Visibly the maps $\Phi \to \varphi_\Phi$ and $\varphi \to \Phi_\varphi$ are mutual inverses. *Naturality* of the isomorphism sending $\varphi \to \Phi_\varphi$ and $\Phi \to \varphi_\Phi$ asserts the commutativity of diagrams attached to $f : A' \to A$ and $g : X' \to X$ and[4] That is, we must have a commutative diagram[5]

$$\begin{tikzcd}
\mathrm{Hom}(A \otimes X, B) \arrow[r, "\approx"] \arrow[d, "h\circ(-)\circ(f\otimes g)"'] & \mathrm{Hom}(A, \mathrm{Hom}(X, B)) \arrow[d, "a'\to\left(x'\to h(((-)(fa'))(gx'))\right)"] \\
\mathrm{Hom}(A' \otimes X, B') \arrow[r, "\approx"'] & \mathrm{Hom}(A', \mathrm{Hom}(X', B'))
\end{tikzcd}$$

[4] Yes, the order of the primed and unprimed symbols is opposite for $h : B \to B'$.

[5] The algebraic notation for the induced maps is awkward and barely intelligible, while it is clear in terms of diagrams. There is no choice about what the maps must be, while the complexity of notating them exceeds that of the underlying reality.

Note the awkwardness on the right-hand side of the diagram, driving us to the more verbose description.

This is easy to check: starting with Φ in the upper left, going down gives $\Phi' = h \circ \Phi \circ (f \otimes g)$, and then going to the right gives φ' such that

$$\varphi'(a')(x') \;=\; \Phi'(a' \otimes x') \;=\; (h \circ \Phi \circ (f \otimes g)(a' \otimes x') \;=\; h\Phi(fa' \otimes gx')$$

Going the other way around the diagram, first we obtain φ such that $\varphi(a)(x) = \Phi(a \otimes x)$. Going down the right side gives φ' such that

$$\varphi'(a')(x') \;=\; \varphi(fa')(gx') \;=\; h\Phi(fa' \otimes gx')$$

The two outcomes are the same, which is the naturality. ///

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3. *Yoneda's lemma*

The innocent-seeming property of $\mathrm{Hom}(X, -)$ below is a special case of **Yoneda's Lemma**. [6]

[3.0.1] **Theorem:** We have *sufficient* criteria for exactness: given a sequence $0 \to A \to B \to C \to 0$,

$$\mathrm{Hom}(X, A) \xrightarrow{f\circ -} \mathrm{Hom}(X, B) \xrightarrow{g\circ -} \mathrm{Hom}(X, C) \quad \text{exact for all } X \quad \Longrightarrow \quad A \xrightarrow{f} B \xrightarrow{g} C \quad \text{exact}$$

Similarly,

$$\mathrm{Hom}(C, X) \xrightarrow{-\circ g} \mathrm{Hom}(B, X) \xrightarrow{-\circ f} \mathrm{Hom}(A, X) \quad \text{exact for all } X \quad \Longrightarrow \quad A \xrightarrow{f} B \xrightarrow{g} C \quad \text{exact}$$

[3.0.2] **Remark:** Exactness of $A \to B \to C$ does *not* imply exactness of the Hom diagram for all X. This was visible in proving *left* exactness of $M \to \mathrm{Hom}(M, X)$.

Proof: On one hand, with $X = A$ and $F : X \to A$ the identity, exactness of the Hom sequence implies

$$0 \;=\; g \circ f \circ F \;=\; g \circ f$$

so $\mathrm{Im} f \subset \ker g$. On the other hand, with $X = \ker g$ and $F : X \to B$ the inclusion, exactness of the Hom sequence (with $g \circ F = 0$) implies that there is $F' : X \to A$ such that $f \circ F' = F$. Then

$$\ker g = \mathrm{Im} F = \mathrm{Im}(f \circ F') \subset \mathrm{Im} f$$

Putting the two containments together gives $\ker g = \mathrm{Im} f$. This proves the result for the covariant Hom functor.

For the contravariant Hom functor $M \to \mathrm{Hom}(M, X)$, with $X = C$ and $F : C \to X$ the identity, the exactness of the Hom sequence gives

$$0 = F \circ g \circ f = g \circ f$$

Thus, $\mathrm{Im} f \subset \ker g$. On the other hand, with $X = B/\mathrm{Im} f$ and $F : B \to X$ the quotient map, by exactness of the Hom sequence there is $F' : C \to X$ such that $F' \circ g = F$. Thus, the kernel of g cannot be larger than $\mathrm{Im} f$, or $F : B \to B/\mathrm{Im} f$ could not factor through it. Thus, we have exactness. ///

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[6] A functor $X \to \mathrm{Hom}(X, -)$ from a category whose hom sets $\mathrm{Hom}(A, B)$ are abelian groups, to the category of abelian groups, is an instance of a *Yoneda imbedding.*

4. *Half-exactness of adjoint functors*

A functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ on categories of modules is **additive**[7] when

$$F(A \oplus B) \approx FA \oplus FB \qquad \text{(for all } A, B \in \mathscr{C}\text{)}$$

As usual, this isomorphism must be *natural*.

Functors $L : \mathscr{C} \to \mathscr{D}$ and $R : \mathscr{D} \longrightarrow \mathscr{C}$ are **adjoint** when there is a *natural* isomorphism

$$\mathrm{Hom}(LA, B) \approx \mathrm{Hom}(A, RB) \qquad \text{(for every } A, B\text{)}$$

An *additive* functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is *left-exact* when[8]

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad \text{exact} \qquad \Longrightarrow \qquad 0 \longrightarrow FA \longrightarrow FB \longrightarrow FC \quad \text{exact}$$

and *right-exact* when

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad \text{exact} \qquad \Longrightarrow \qquad FA \longrightarrow FB \longrightarrow FC \longrightarrow 0 \quad \text{exact}$$

[4.0.1] Theorem: Let L, R be mutually adjoint *additive* functors, with L the left and R the right adjoint. Then L is right half-exact and R is left half-exact. That is,

$$0 \to A \to B \to C \to 0 \quad \text{exact} \qquad \Longrightarrow \qquad LA \to LB \to LC \to 0 \quad \text{exact}$$

and

$$0 \to A \to B \to C \to 0 \quad \text{exact} \qquad \Longrightarrow \qquad 0 \to RA \to RB \to RC \quad \text{exact}$$

Proof: Left exactness of $M \to \mathrm{Hom}(X, M)$ for any X applies to X replaced by LX, so

$$0 \longrightarrow \mathrm{Hom}(LX, A) \longrightarrow \mathrm{Hom}(LX, B) \longrightarrow \mathrm{Hom}(LX, C) \qquad \text{(exact)}$$

By adjointness of L and R, and *naturality* of the adjunction isomorphisms, we have a commutative diagram with exact top row,

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}(LX, A) & \longrightarrow & \mathrm{Hom}(LX, B) & \longrightarrow & \mathrm{Hom}(LX, C) \\
 & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx \\
0 & \longrightarrow & \mathrm{Hom}(X, RA) & \longrightarrow & \mathrm{Hom}(X, RB) & \longrightarrow & \mathrm{Hom}(X, RC)
\end{array}$$

[7] More abstractly, a common structure on a category $\mathscr{C}$ is that it be a **pre-additive** category, meaning that for $A, B \in \mathscr{C}$ the hom set $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an *abelian group*, and the composition

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(A, C)$$

is *bilinear*. Thus, more generally, for pre-additive categories $\mathscr{C}, \mathscr{D}$, a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is *additive* when it preserves coproducts. In contemporary use, a category is **additive** when it is pre-additive *and* has a zero object, has finite coproducts. The pre-additive structure causes finite coproducts to be *products*, as well.

[8] This is for *covariant* functors.

Then the bottom row is exact, for all X. By Yoneda's lemma,

$$0 \longrightarrow RA \longrightarrow RB \longrightarrow RC \qquad\qquad \text{(exact)}$$

Similarly, for the other Hom functor, when in a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(C, RX) & \longrightarrow & \mathrm{Hom}(B, RX, B) & \longrightarrow & \mathrm{Hom}(A, RX) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}(LC, X) & \longrightarrow & \mathrm{Hom}(LB, X) & \longrightarrow & \mathrm{Hom}(LC, X) \end{array}$$

the *top* row is exact, then the bottom row is exact. When this holds for all X, by Yoneda

$$LA \longrightarrow LB \longrightarrow LC \longrightarrow 0 \qquad\qquad \text{(exact)}$$

noting that this second Hom functor $M \to \mathrm{Hom}(M, X)$ is *contravariant*. ///

[4.0.2] Corollary: The natural (adjunction) isomorphism $\mathrm{Hom}(A \otimes X, B) \approx \mathrm{Hom}(A, \mathrm{Hom}(X, B))$ yields the *right* exactness of $M \to M \otimes X$. ///