Solutions 6

#1 Fix an element h of a group G. Define $f : G \to G$ by $f(g) = hgh^{-1}$. Prove that f is an automorphism of G.

First, let's show that this is a group homomorphism. Indeed,

$$f(gg') = h(gg')h^{-1} = hgh^{-1} \cdot hg'g^{-1} = f(g) \cdot f(g')$$

Thus, f is a homomorphism. Next, let's prove that f is surjective: that is, given $g \in G$, find g' so that $f(g') = g$. Looking at the definition of the function f, we can anticipate that $g' = h^{-1}gh$ will work. Checking:

$$f(g') = f(h^{-1}gh) = h(h^{-1}gh)h^{-1} = g$$

And check that f is injective: suppose that $f(g) = f(g')$. Then $hgh^{-1} = hg'h^{-1}$. Right multiplying by h and left multiplying by h^{-1}, we obtain $g = g'$, so f is injective. Thus, altogether, f is an isomorphism.

#2 Show that any group of order 35 is cyclic.

Assume that the group G of order 35 is not cyclic. Then there is no element of order 35. By Lagrange, the only possible orders of elements are 1,5,7. As discussed in class, each subgroup of order 5 contains $5-1$ elements of order 5, and two distinct subgroups of order 5 have no elements of order 5 in common (by Lagrange's theorem again), since 5 is prime. The same applies to subgroups and elements of order 7, since 7 is prime. Invoking Sylow's theorem, for some non-negative integers a, b the number of subgroups of order 5 is $5a+1$ and the number of subgroups of order 7 is $7b+1$. Then, counting the elements of G two ways,

$$35 = 1 + (5a+1)(5-1) + (7b+1)(7-1)$$

Simplifying, this is

$$24 = 20a + 42b$$

Since $42 > 24$, it must be that $b = 0$. But then $24 = 20a$ is impossible, since 20 doesn't divide 24. Thus, there must have been an element of order 35 in G, so G is cyclic.

#3 List the 8 abelian groups of order 900.

Invoke the Structure Theorem for finite abelian groups: find all tuples of integers $d_1, d_2, \ldots, d_n$ all bigger than 1, whose product is 900, and so that $d_i | d_{i+1}$ for all indices i. Invoking Sun Ze's theorem, it suffices to solve this problem for the maximal prime powers 2^2, 3^2, 5^2 occurring in 900, and then combine the results. Quite generally, for prime p there are exactly two abelian groups of order p^2: $\mathbf{Z}/p^2$ and $\mathbf{Z}/p \oplus \mathbf{Z}/p$, corresponding to the two sequences of elementary divisors (p^2) and (p, p). Thus, in the case at hand, there are two choices for the 2-part of the group, 2 choices for the 3-part, and 2 choices for the 5-part. Combining and listing these, we have $8 = 2^3$ abelian groups: $\mathbf{Z}/900$, $\mathbf{Z}/2 \oplus \mathbf{Z}/450$, $\mathbf{Z}/3 \oplus \mathbf{Z}/300$, $\mathbf{Z}/5 \oplus \mathbf{Z}/180$, $\mathbf{Z}/6 \oplus \mathbf{Z}/150$, $\mathbf{Z}/10 \oplus \mathbf{Z}/90$, $\mathbf{Z}/15 \oplus \mathbf{Z}/60$, $\mathbf{Z}/30 \oplus \mathbf{Z}/30$.