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Exponentiating matrices

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For an n-by-n complex matrix M, the series for the exponential function $M \to e^M$ is

$$M \longrightarrow \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

To prove absolute convergence, since all topologies on a finite-dimensional complex vector space are the same [1] , it suffices to prove absolute convergence in *operator* norm

$$\|T\|_{\mathrm{op}} = \sup_{\|x\| \le 1} \|Tx\|$$

where $\|x\|$ is the usual norm on $\mathbb{C}^n$. The utility of this choice is the obvious sub-multiplicativity

$$\|T^n\|_{\mathrm{op}} \le \|T\|_{\mathrm{op}}^n$$

Then the *idea*[2] is that

$$\|\sum_{n=0}^{\infty} \frac{M^n}{n!}\|_{\mathrm{op}} \le \sum_{n=0}^{\infty} \frac{\|M^n\|_{\mathrm{op}}}{n!} \le \sum_{n=0}^{\infty} \frac{\|M\|_{\mathrm{op}}^n}{n!} < +\infty$$

by the usual convergence of the exponential series with complex arguments. More scrupulously, to prove convergence of an infinite sum we should estimate the *finite tails*: for $n_1 \le n_2$,

$$\|\sum_{n=n_1}^{n_2} \frac{M^n}{n!}\|_{\mathrm{op}} \le \sum_{n=n_1}^{n_2} \frac{\|M^n\|_{\mathrm{op}}}{n!} \le \sum_{n=n_1}^{n_2} \frac{\|M\|_{\mathrm{op}}^n}{n!}$$

That is, the finite tails of the matrix-valued exponential series are dominated in operator norm by the finite tails of the exponential series with complex arguments, which go to 0 appropriately. ///

[1] That is, there is a unique topology on a finite-dimensional vector space over a complete, non-discrete division algebra, for which scalar multiplication and vector addition are continuous, and for which points are closed. This assertion is somewhat different from comparisons of *norms* on finite-dimensional spaces.

[2] This is not exactly a proof, because, if taken too literally, might mislead a naive reader. Specifically, one might worry about cancellation in the first sum inside the operator norm. But, more to the point, if convergence hasn't been established, then from an elementary viewpoint it doesn't make sense to talk about the operator norm of the infinite sum at all. It might be less hazardous to simply write

$$\sum_{n=0}^{\infty} \frac{\|M^n\|_{\mathrm{op}}}{n!} \le \sum_{n=0}^{\infty} \frac{\|M\|_{\mathrm{op}}^n}{n!} < +\infty$$

obviously indicating how the formally correct argument will proceed.