(*March 12, 2020*)

02. Harmonic analysis of dihedral groups

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1. *Dihedral groups*

[1.1] Dihedral groups as symmetries of n-gons The dihedral group G is the symmetry group of a regular n-gon. To count the elements of the group, choose a vertex v of the n-gon, and an adjacent vertex w. A symmetry g is completely determined by the image gv, which can be any other vertex, and by gw, which can be either one of the two vertices adjacent to gv. Thus, indeed, there are $2n$ symmetries.

The *rotations* are the symmetries preserving the (cyclic) ordering of vertices. Thus, a rotation g is determined by the image gv, so the subgroup N of rotations has n elements. A *reflection* is an order-2 symmetry *reversing* the ordering of vertices. Imbedding the n-gon in $\mathbb{R}^2$, there are n axes through which the n-gon can be reflected, so there are n reflections. Since $|G| = 2n$, every symmetry is either a rotation or a reflection.

Characterizing rotations x as preserving the ordering of vertices, and reflections s as reversing the ordering, the conjugate sxs^{-1} preserves the ordering, so is a rotation. Thus, the subgroup N of rotations is *normal*.

For any reflection s and rotation x, we claim that $sxs^{-1} = x^{-1}$. First, sxs^{-1} is a rotation, so to determine it it suffices to see how far any given vertex is moved. Since the rotation subgroup is *cyclic*, it suffices to prove this for the rotation x moving vertices by $+1$ in the cyclic order. Let v be a vertex fixed by s. In the composition $xsxs^{-1}$, successively (i) $s = s^{-1}$ doesn't move v (ii) x moves v to $v+1$ (iii) s moves $v+1$ to $v-1$, since s fixes v and reverses the ordering (iv) x moves $v-1$ to v. That is, $xsxs^{-1}$ is a rotation fixing v, so is the identity.

[1.2] Dihedral groups as semi-direct products A semi-direct product description of the dihedral group G is useful for computations. It is a *non-abelian* group fitting into a short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow \{1, \overline{\sigma}\} \longrightarrow 1 \qquad \text{(with } N \text{ cyclic of order } n\text{)}$$

Further, the order-2 group $\{1, \overline{\sigma}\}$ has a *section* $\{1, \overline{\sigma}\} \to G$, and we fix a choice, which specifies a reflection σ, the image of $\overline{\sigma}$. The *non-abelian-ness* requires that

$$\sigma x \sigma^{-1} \;=\; x^{-1} \qquad \text{(for all } x \in N\text{)}$$

That is, G is a non-abelian semi-direct product of normal subgroup N by the group $\{1, \sigma\}$, with relation $\sigma x \sigma^{-1} = x^{-1}$. The dihedral group G has a coset decomposition

$$G \;=\; N \sqcup \sigma N \;=\; N \sqcup N\sigma$$

[1.3] Remark: In fact, most of the following discussion applies to semi-direct products G of *arbitrary* finite abelian groups N by order-2 groups $\{1, \sigma\}$ with the relations $\sigma x \sigma^{-1} = x^{-1}$ for all $x \in N$.

[1.4] Scarcity of characters on dihedral groups Recall that, for finite *abelian* groups A, complex-valued functions f on A admit Fourier expansions

$$f \;=\; \sum_{\chi \in \widehat{A}} c_\chi \cdot \chi \qquad \text{(with } c_\chi \in \mathbb{C}\text{, group homomorphisms } \chi : A \to \mathbb{C}^\times\text{)}$$

In particular, the characters $\chi \in \widehat{A}$ *separate points*, so the number of group homomorphisms $A \to \mathbb{C}^\times$ is $|A|$.

In contrast, dihedral groups G have *few* group homomorphisms to $\mathbb{C}^\times$. Indeed, group homomorphisms $\varphi : G \to \mathbb{C}^\times$ are trivial on *commutators* $ghg^{-1}h^{-1}$, since

$$\varphi(ghg^{-1}h^{-1}) \;=\; \varphi(g)\,\varphi(h)\,\varphi(g)^{-1}\,\varphi(h)^{-1} \;=\; 1 \qquad\qquad (\text{for } g, h \in G)$$

Then, for a group homomorphism $\varphi : G \to \mathbb{C}^\times$,

$$1 \;=\; \varphi(x\sigma x^{-1}\sigma^{-1}) \;=\; \varphi(x \cdot x) \;=\; \varphi(x^2) \qquad\qquad (\text{for } x \in N)$$

That is, these homomorphisms are all trivial on squares of rotations, so there is no hope that arbitrary functions on a dihedral group could be expressed as linear combinations of homomorphisms $G \to \mathbb{C}^\times$.

[1.5] Restriction to N

Although dihedral groups G have too few group homomorphisms to $\mathbb{C}^\times$ to allow expansion of arbitrary functions on G, the group G is not so far away from its abelian subgroup N:

$$G \;=\; N \sqcup \sigma N$$

We forget about the action of the reflection σ for a little while, and restrict attention to the right translation action [1] of the subgroup N on functions on the whole group G.

Both cosets N and σN are stable under right multiplication by N, inside G, and N acts on functions f on G by

$$(x \cdot f)(g) \;=\; f(gx) \qquad\qquad (\text{for } x \in N \text{ and } g \in G)$$

Let $L^2(X)$ be the complex-valued functions on a finite set X. The subspaces $L^2(N)$ and $L^2(\sigma N)$ of $L^2(G)$ are stable under the right translation action of N, and every function on G is the sum of its restrictions to N and σN, so

$$L^2(G) \;=\; L^2(N) \oplus L^2(\sigma N) \qquad\qquad (\text{respecting the action of } N)$$

Taking advantage of the specifics, we can write down many functions on G in terms of group homomorphisms $\psi : N \to \mathbb{C}^\times$ of N:

$$f(g) \;=\; \begin{cases} \sum_\psi a_\psi\, \psi(x) & (\text{for } g = x \in N,\ a_\psi \in \mathbb{C}) \\ \\ \sum_\psi b_\psi\, \psi(x) & (\text{for } g = \sigma x \in \sigma N,\ x \in N,\ b_\psi \in \mathbb{C}) \end{cases}$$

Recall from above that there are exactly $|N|$ characters $\psi \in \widehat{N}$, for finite abelian N, and these characters are linearly independent functions on N. Thus, the dimension of the complex vector space of functions with such expressions is $2 \cdot |N|$, which is $|G|$, the dimension of $L^2(G)$. By dimension-counting, there is no room for other functions in $L^2(G)$: every function in $L^2(G)$ is of this form.

Thus, when we forget the reflection σ, functions on G *do* have a Fourier expansion in terms of the normal *abelian* subgroup N.

[1.6] Reintroduction of the reflection σ

Now let σ act on the right on G, thereby acting on functions on G, analyzed in terms of the two-part Fourier expansion in terms of N.

The two cosets N and σN are interchanged by right multiplication by σ, but elements σx and x (with $x \in N$) are not simply interchanged. Rather, there is an additional inversion:

$$x \cdot \sigma \;=\; \sigma \cdot \sigma x \sigma \;=\; \sigma \cdot x^{-1} \qquad\qquad (\text{for } x \in N)$$

and

$$\sigma x \cdot \sigma \;=\; \sigma x \sigma \;=\; x^{-1} \qquad\qquad (\text{for } x \in N)$$

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[1] Right translation action of G on $L^2(G)$ is also called the *right regular representation* of G.

Thus, noting that $\psi(x^{-1}) = \psi(x)^{-1} = \overline{\psi}(x)$, with

$$f(g) \;=\; \begin{cases} \sum_\psi a_\psi\, \psi(x) & (\text{for } g = x \in N) \\ \\ \sum_\psi b_\psi\, \psi(x) & (\text{for } g = \sigma x \in \sigma N) \end{cases}$$

we have

$$f(g\sigma) \;=\; \begin{cases} \sum_\psi a_\psi\, \overline{\psi}(x) & (\text{for } g = \sigma x \in \sigma N) \\ \\ \sum_\psi b_\psi\, \overline{\psi}(x) & (\text{for } g = x \in N) \end{cases} \;=\; \begin{cases} \sum_\psi b_{\overline{\psi}}\, \psi(x) & (\text{for } g = x \in N) \\ \\ \sum_\psi a_{\overline{\psi}}\, \psi(x) & (\text{for } g = \sigma x \in \sigma N) \end{cases}$$

Thus, the coefficients are interchanged, and, additionally, twisted by complex-conjugation of characters.

[1.7] Remark: Most characters $\psi : N \to \mathbb{C}^\times$ are not equal to their complex conjugates $\overline{\psi}$, but a few are. The trivial character obviously is equal to its conjugate. For $|N|$ *even*, there is a non-trivial ± 1-valued character, which is therefore equal to its conjugate.

[1.8] Minimal G-stable spaces of functions Looking at the transformation rule for N-Fourier expansions of functions on G under right translation by the reflection σ, we can observe some small G-stable subspaces, as follows. For brevity, for $\psi \in \widehat{N}$, let

$$f_1^\psi(g) \;=\; \begin{cases} \psi(x) & (\text{for } g = x \in N) \\ \\ 0 & (\text{for } g \in \sigma N) \end{cases}$$

$$f_2^\psi(g) \;=\; \begin{cases} 0 & (\text{for } g \in N) \\ \\ \overline{\psi}(x) & (\text{for } g = \sigma x \in \sigma N) \end{cases}$$

and let

$$I_\psi \;=\; \mathbb{C}\cdot f_1^\psi + \mathbb{C}\cdot f_2^\psi$$

Right translation by σ interchanges f_1^ψ and f_2^ψ, and N acts on f_1^ψ by ψ, and on f_2^ψ by $\overline{\psi}$. That is, for emphasis, I_ψ is G-stable. [2] For abelian groups, minimal stable subspaces are always one-dimensional. For dihedral groups G, for $\psi \neq \overline{\psi}$, the *two-dimensional* space I_ψ is *minimal*. To see this, take

$$f \;=\; af_1^\psi + bf_2^\psi \qquad\qquad (\text{with } a, b \in \mathbb{C}, \text{ not both } 0)$$

The action of $x \in N$ on f gives

$$x\cdot f \;=\; \psi(x)\cdot a\cdot f_1^\psi + \overline{\psi}(x)\cdot b\cdot f_2^\psi$$

The image $\sigma\cdot f = bf_1^\psi + af_2^\psi$ is already linearly independent of f, proving minimality of I_ψ unless $a = \pm b$. With $a = \pm b$, when $\psi \neq \overline{\psi}$, take x such that $\psi(x) \neq \overline{\psi}(x)$, and then f and $x\cdot f$ are linearly independent since neither of a, b is 0. This proves that I_ψ is a minimal G-stable subspace of $L^2(G)$ for $\psi \neq \overline{\psi}$.

When $\psi = \overline{\psi}$, there are smaller G-stable subspaces of I_ψ suggested by the previous discussion, namely

$$\mathbb{C}\cdot(f_1^\psi + f_2^\psi) \qquad \text{and} \qquad \mathbb{C}\cdot(f_1^\psi - f_2^\psi)$$

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[2] Also, I_ψ is the collection of functions $f \in L^2(G)$ such that $f(xg) = \psi(x)\cdot f(g)$ for $x \in N$ and $g \in G$. This is *left equivariance by* N, ψ. Spaces of functions formed in this way have special significance, explainable only in somewhat more sophisticated settings. Nevertheless, keywords should be mentioned: these are standard models of *induced representations*, and thereby appear in *Frobenius reciprocity.* The latter is one of the most significant instances of an *adjoint functor* relation. These things are important, but understandably not-so-easy to understand in a relatively elementary context.

[1.9] Remark: For abelian groups A, the minimal translation-stable subspaces of $L^2(A)$ are *one-dimensional*, consisting of scalar multiples $\mathbb{C}\cdot\chi$ of characters $\chi : A \to \mathbb{C}^\times$. In contrast, minimal stable subspaces of $L^2(G)$ for dihedral groups G are mostly *two-dimensional*. Further, the parametrizing scheme for the minimal stable subspaces for dihedral groups is less clear: for $\psi \in \widehat{N}$ with $\psi \neq \overline{\psi}$, the G-stable subspace I_ψ is minimal, but for $\psi = \overline{\psi}$ the space I_ψ is a direct sum of two one-dimensional G-stable subspaces. [3]

In particular, the standard notion of *simultaneous eigenvector* is insufficient to accommodate the decomposition of $L^2(G)$ into minimal G-stable subspaces. As the above discussion suggest, the notion of eigenvector is replaced by **minimal stable subspace**.

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2. *Products on dihedral groups*

In the simple scenario of group homomorphisms $\alpha, \beta : A \to \mathbb{C}^\times$ of an abelian group A, the product $\alpha\cdot\beta$ is again a group homomorphism. In other words, the product of two simultaneous eigenfunctions [4] for the translation action of A on $L^2(A)$ is itself a simultaneous eigenfunction.

For non-abelian groups, as observed above for dihedral groups G, the notion of simultaneous eigenvector is itself already insufficient, since the minimal stable subspaces of $L^2(G)$ are mostly *two-dimensional*, not one-dimensional.

Thus, even the simplest non-abelian groups require a more sophisticated viewpoint, introduced below. [5]

[2.1] Representations For a finite group G, a group homomorphism $G \to \mathrm{Aut}_{\mathbb{C}} V$ of G to the group of $\mathbb{C}$-linear automorphisms of a complex vector space V is called a *representation* of G on V, and V is the *representation space*. Write the action of $g \in G$ on $v \in V$ simply as [6]

$$g \times v \longrightarrow g\cdot v = gv$$

The prototype of a representation is the right-translation action of G on the complex-valued functions $L^2(G)$ on G.

A *homomorphism* of G-representations V, W, or simply *G-homomorphism*, is a $\mathbb{C}$-vector-space map $\varphi : V \to W$ which *respects* the action of G, in the sense that

$$\varphi(g\cdot v) \;=\; g\cdot\varphi(v) \qquad \text{(for all } g\in G \text{ and } v \in V)$$

Important examples of G-homomorphisms are *inclusions* of G-stable subspaces, and *quotients* by G-stable subspaces.

[2.2] Direct sums of representations For two G-representations V, W, the direct sum $V \oplus W$ is the vector space direct sum, with the G-action

$$g\cdot(v\oplus w) \;=\; gv \oplus gw$$

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[3] For trivial ψ, the two one-dimensional subspaces are the trivial one (in which the whole group acts trivially), and that in which rotations act trivially, while reflections act by -1. Similarly, for non-trivial ± 1-valued ψ, there is the one-dimensional subspace in which reflections act trivially, and that in which reflections act by -1.

[4] When the vector space consists of *functions* on a set, then eigenvectors for operators are often called *eigenfunctions*.

[5] Specifically, while *sums* of functions are adequately described via *direct sums* of G-stable spaces, description of *products* of functions on non-abelian groups requires *tensor products*.

[6] Sometimes the homomorphism $G \to \mathrm{Aut}_{\mathbb{C}} V$ of G is *named*, a typical default being $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)$. Then the action is written $\pi(g)(v)$, rather than gv. This can be a useful disambiguation, but is heavier notation.

For example, for an abelian group A, the Fourier expansion assertion

$$L^2(A) \approx \bigoplus_{\psi\in\widehat{A}} \mathbb{C}\cdot\psi \qquad \text{(as } A\text{-representation spaces, for } A \text{ finite abelian)}$$

says that $L^2(A)$ is the direct sum of the one-dimensional representation spaces $\mathbb{C}\cdot\psi$, on which A acts by ψ.

[2.3] Irreducibles A G-representation V is *irreducible* if it has no G-stable subspaces other than $\{0\}$ and V itself. This is a correct extension of the notion of *eigenvalue* to a possibly-non-abelian setting.

Inside a representation V of G, *minimal G-stable* subspaces are *irreducible*. Conversely, irreducible subspaces are minimal G-stable subspaces.

For *finite abelian G*, elementary spectral theory implies that all irreducibles are *one-dimensional*. Certainly any one-dimensional representation space is irreducible, but, as dihedral groups illustrate, non-abelian groups have higher-dimensional irreducibles.

To express a given representation V of G as a direct sum of irreducibles is to *decompose* V. This is a correct general version of expression of a given vector as a linear combination of eigenvectors.

For example, whether or not G is abelian, one-dimensional spaces $\mathbb{C}\cdot\psi$ spanned by group homomorphism $\psi : G \to \mathbb{C}^\times$ are always irreducible. For *abelian A*, there are sufficiently many of these one-dimensional irreducibles so that $L^2(A)$ is *decomposed*: again,

$$L^2(A) \approx \bigoplus_{\psi\in\widehat{A}} \mathbb{C}\cdot\psi \qquad \text{(as } A\text{-representation spaces, for } A \text{ finite abelian)}$$

[2.4] Remark: It is a traditional source of cognitive dissonance that A acts on $\mathbb{C}\cdot\psi$ by ψ, since the basis element ψ is identical with the group homomorphism $A \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}\cdot\psi)$: for any constant $c \in \mathbb{C}$, the function $c\cdot\psi$ is in $\mathbb{C}\cdot\psi$, and

$$\big(a\cdot(c\psi)\big)(b) = (c\psi)(ba) = \psi(a)\cdot c\psi(b) = \Big(\psi(a)\cdot c\psi\Big)(b) \qquad \text{(for } a, b \in A)$$

This holds for all b, so $a\cdot(c\psi) = \psi(a)\cdot c\psi$. That is, for finite abelian groups, the *group homorphism $G \to \mathbb{C}^\times$* is also a basis vector for the *representation space*.

The discussion of functions on dihedral groups G above *decomposes* $L^2(G)$ into irreducibles, most of which are two-dimensional. As above, let

$$f_1^\psi(g) = \begin{cases} \psi(x) & \text{(for } g = x \in N) \\ 0 & \text{(for } g \in \sigma N) \end{cases}$$

$$f_2^\psi(g) = \begin{cases} 0 & \text{(for } g \in N) \\ \overline{\psi}(x) & \text{(for } g = \sigma x \in \sigma N) \end{cases}$$

and

$$I_\psi = \mathbb{C}\cdot f_1^\psi + \mathbb{C}\cdot f_2^\psi$$

Then for dihedral $G = N \sqcup \sigma N$, letting $\psi \in \widehat{N}$,

$$L^2(G) = \bigoplus_{\psi\neq\overline{\psi}} I_\psi \oplus \bigoplus_{\psi=\overline{\psi}} \mathbb{C}(f_1^\psi + f_2^\psi) \oplus \bigoplus_{\psi=\overline{\psi}} \mathbb{C}(f_1^\psi - f_2^\psi)$$

That is, the sum over $\psi \neq \overline{\psi}$ is a sum of *two-dimensional* irreducibles I_ψ, while the smaller sum over $\psi = \overline{\psi}$ involves only one-dimensional irreducibles.

[2.5] Decomposition into irreducibles In very general circumstances, representations of groups may *fail* to be direct sums of irreducibles, and this failure can be awkward. Fortunately, we will see that *finite-dimensional* (complex) representations of *finite* groups are direct sums of irreducibles. [7] Let G be an arbitrary finite group. Given a G-representation V, we claim that there is a hermitian inner product $\langle , \rangle$ on V for which G acts by *unitary operators*, that is,

$$\langle gv, gw \rangle \;=\; \langle v, w \rangle \qquad\qquad (\text{for all } v, w \in V,\; g \in G)$$

Indeed, start with *any* hermitian inner product $\langle , \rangle_o$ on V, and *average*:

$$\langle v, w \rangle \;=\; \sum_{g \in G} \langle gv, gw \rangle_o$$

With this in hand, given a G-stable subspace W of V, the orthogonal complement $W^\perp$ is also G-stable:

$$\langle gv, w \rangle \;=\; \langle v, g^{-1} w \rangle \;=\; 0 \qquad\qquad (\text{for } w \in W,\; v \in W^\perp)$$

Thus, either V is irreducible, or it has a proper G-subrepresentation W. The orthogonal complement $W^\perp$ is also a G-subrepresentation, and $V = W \oplus W^\perp$. By induction on dimension, W and $W^\perp$ decompose into irreducibles. Thus, V has such a decomposition.

That is, finite-dimensional complex representations of finite groups do decompose into irreducibles. [8]

[2.6] Decomposition of representations of dihedral groups The relatively large abelian subgroup N of the dihedral G gives a convenient parametrization of irreducibles of G inside $L^2(G)$. In fact, the same ideas apply to *all* (finite-dimensional) representations of G, as follows.

Let V be a finite-dimensional representation of dihedral $G = N \sqcup \sigma N$. Forgetting the action of the reflection σ for a moment, V is a finite-dimensional representation for the *abelian* group N, so (see the appendix) V is a direct sum of ψ-eigenspaces in a more elementary sense, as ψ ranges over group homomorphisms $\psi : N \to \mathbb{C}^\times$:

$$V \;=\; \bigoplus_\psi V_\psi \qquad\qquad (x \cdot v = \psi(x) \cdot v \text{ for } v \in V_\psi, \text{ for all } x \in N)$$

Just as in the concrete case of $L^2(G)$, the action of σ maps V_ψ to $V_{\overline{\psi}}$:

$$x \cdot (\sigma \cdot v) \;=\; \sigma \cdot (\sigma x \sigma \cdot v) \;=\; \sigma \cdot (\psi(\sigma x \sigma) \cdot v) \;=\; \sigma \cdot (\psi(x^{-1}) \cdot v) \;=\; \overline{\psi}(x) \cdot (\sigma \cdot v) \qquad\qquad (\text{for } v \in V_\psi)$$

Since $\sigma^2 = 1$, the map $V_\psi \to V_{\overline{\psi}}$ is a vector-space isomorphism.

For $\psi \neq \overline{\psi}$, for any $0 \neq v \in V_\psi$, the two-dimensional subspace

$$\mathbb{C} \cdot v + \mathbb{C} \cdot \sigma v \;\subset\; V_\psi + V_{\overline{\psi}}$$

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[7] A group representation is said to be *completely reducible* if it is a direct sum of irreducibles.

[8] The question of *uniqueness* of decomposition into irreducibles naturally arises. Already for abelian groups it is easy to be too naive: eigenvalues can occur with multiplicities greater than 1, and uniqueness even in the abelian case only applies to decomposition into eigen*spaces*, not eigen*vectors*. For non-abelian groups, *eigenvector* becomes approximately *irreducible representation*, and *eigenspace* becomes *isotype*: for an irreducible ρ of G, the ρ-*isotype* in a G-representation V is the (not direct!) sum of all isomorphic copies of ρ inside V.

is G-stable. The argument used earlier for the subspace $I_\psi = \mathbb{C}f_1^\psi + \mathbb{C}f_2^\psi$ proves the *irreducibility* of the two-dimensional space $\mathbb{C}v + \mathbb{C}\sigma v$, as follows. Given $v' = av + b\sigma v$ with $a, b \in \mathbb{C}$, already $\sigma v'$ and v' are linearly independent unless $a = \pm b$. So consider $a = \pm b$. Using $\psi \neq \overline{\psi}$, let $x \in N$ such that $\psi(x) \neq \overline{\psi}(x)$. Then

$$x \cdot v' \; = \; \psi(x)v + \overline{\psi}(x)\sigma v$$

Since $\psi(x) \neq \overline{\psi}(x)$, the two vectors v' and xv' are linearly independent. That is, any G-stable non-zero subspace of this two-dimensional space is the whole, proving its irreducibility.

Now we show that every two-dimensional irreducible is isomorphic to one of those we have already constructed. The linear map

$$\varphi \; : \; af_1^\psi + bf_2^\psi \; \longrightarrow \; av + b\sigma v \qquad \text{(for } a, b \in \mathbb{C}\text{)}$$

respects the action of G, that is, φ is a G-isomorphism from the two-dimensional irreducible $I_\psi = \mathbb{C}f_1^\psi + \mathbb{C}f_2^\psi$ inside $L^2(G)$ to the abstract irreducible $\mathbb{C}v + \mathbb{C}\sigma v$. This is easy to verify: for $x \in N$,

$$\varphi(x \cdot (af_1^\psi + bf_2^\psi)) \; = \; \varphi(\psi(x)af_1^\psi + \overline{\psi}(x)bf_2^\psi) \; = \; \psi(x)a\varphi(f_1^\psi) + \overline{\psi}(x)b\varphi(f_2^\psi)$$
$$= \; \psi(x)av + \overline{\psi}(x)b\sigma v \; = \; x \cdot (av + b\sigma v) \; = \; x \cdot \varphi(af_1^\psi + bf_2^\psi)$$

Regarding σ,

$$\varphi(\sigma \cdot (af_1^\psi + bf_2^\psi)) \; = \; \varphi(af_2^\psi + bf_1^\psi) \; = \; a\varphi(f_2^\psi) + b\varphi(f_1^\psi)$$
$$= \; a\sigma v + bv \; = \; \sigma \cdot (av + b\sigma v) \; = \; \sigma \cdot \varphi(af_1^\psi + bf_2^\psi)$$

It is even easier to see that the one-dimensional irreducibles of the dihedral group occur inside $L^2(G)$.

[2.7] Multiplicities

For an abelian group A, every irreducible appears in $L^2(A)$ exactly once. In contrast, for dihedral groups G, the two-dimensional irreducibles appear in $L^2(G)$ *twice*. [9] For dihedral groups, the only-slightly-surprising point is that, for $\psi \neq \overline{\psi}$, we have a *$G$-isomorphism*

$$I_\psi \; \approx \; I_{\overline{\psi}} \qquad (G\text{-isomorphism})$$

by the flip map

$$af_1^\psi + bf_2^\psi \; \longrightarrow \; bf_1^{\overline{\psi}} + af_2^{\overline{\psi}}$$

In fact, this isomorphism is given by *left* multiplication by σ, observing that

$$f_1^\psi(\sigma \cdot g) \; = \; f_2^{\overline{\psi}}(g) \qquad f_2^\psi(\sigma \cdot g) \; = \; f_1^{\overline{\psi}}(g)$$

For emphasis: the two irreducible G-subrepresentations I_ψ and $I_{\overline{\psi}}$ of $L^2(G)$ are *isomorphic*. Synonymously, the two spaces are *G-isomorphic*.

They are linearly independent, since the functions f_1^ψ, f_2^ψ, $f_1^{\overline{\psi}}$, and $f_2^{\overline{\psi}}$ are visibly linearly independent. Thus, the multiplicity of this two-dimensional irreducible representation in $L^2(G)$ is 2.

[2.8] Products

Functions in $L^2(G)$ can be *multiplied*, and the multiplication map

$$f \times F \; \longrightarrow \; f \cdot F \qquad \text{(for } f, F \in L^2(G)\text{)}$$

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[9] This multiple occurrence of irreducibles in $L^2(G)$ for dihedral groups G is part of a larger pattern: for finite groups G, the number of linearly independent copies of an irreducible ρ in $L^2(G)$ is equal to the dimension of ρ. The number of linearly independent copies is called the *multiplicity* of ρ in the larger representation.

is a $\mathbb{C}$-bilinear map respecting the right-translation G-action in the sense that

$$(g\cdot(fF))(h) \;=\; (fF)(hg) \;=\; f(hg)\,F(hg) \;=\; (g\cdot f)(h)\,(g\cdot F)(h)$$

For *abelian* groups A, every function in $L^2(A)$ is a linear combination of simultaneous eigenfunctions for A, *and* products of eigenfunctions are eigenfunctions. For *non-abelian* groups such as dihedral G, the situation is more complicated.

Parametrize typical irreducibles of G by characters of N: for $\psi \neq \overline{\psi}$, the two-dimensional space I_ψ is irreducible, and $I_\psi \approx I_{\overline{\psi}}$. It is convenient to use the characterization of I_ψ by equivariance under *left* translation by N: we claim that [10]

$$I_\psi \;=\; \{f\in L^2(G)\;:\; f(xg)=\psi(x)\cdot f(g) \text{ for } x\in N, g\in G\}$$

Indeed, on N, the equivariance condition implies

$$f(x) \;=\; f(x\cdot 1) \;=\; \psi(x)\cdot f(1)$$

On σN, the equivariance condition implies

$$f(\sigma x) \;=\; f(\sigma x\sigma\cdot\sigma) \;=\; f(x^{-1}\cdot\sigma) \;=\; \psi(x^{-1})\cdot f(\sigma) \;=\; \overline{\psi}(x)\cdot f(\sigma)$$

This shows that left N,ψ-equivariant f is the linear combination

$$f \;=\; f(1)\cdot f_1^\psi + f(\sigma)\cdot f_2^\psi$$

Conversely, f_1^ψ and f_2^ψ are directly checked to have this left N,ψ-equivariance.

Next, we examine *pointwise multiplication* of functions in irreducibles inside $L^2(G)$. Take two characters $\alpha\neq\beta$ of N, with $\alpha\neq\overline{\alpha}$, $\beta\neq\overline{\beta}$, and $\overline{\alpha}\neq\beta$. The collection

$$I_\alpha\cdot I_\beta \;=\; \{\text{linear combinations of products } f\cdot F, \text{ for } f\in I_\alpha,\; F\in I_\beta\}$$

is a subset of (the standard version of the principal series as just above)

$$\{f\in L^2(G)\;:\; f(xg)=(\alpha\beta)(x)\cdot f(g) \text{ for } x\in N, g\in G\} \;=\; I_{\alpha\beta}$$

Indeed, element-wise,

$$f_1^\alpha\cdot f_1^\beta \;=\; f_1^{\alpha\beta} \qquad\qquad f_2^\alpha\cdot f_2^\beta \;=\; f_2^{\alpha\beta}$$

so

$$I_\alpha\cdot I_\beta \;=\; I_{\alpha\beta} \qquad\qquad (\text{all these being subspaces of } L^2(G))$$

On the other hand, $I_\alpha \approx I_{\overline{\alpha}}$ as G-representations, and

$$I_{\overline{\alpha}}\cdot I_\beta \;=\; I_{\overline{\alpha}\beta}$$

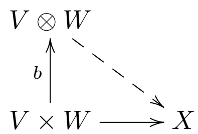
However, $I_{\overline{\alpha}\beta} \not\approx I_{\alpha\beta}$ as (irreducible) G-representations, since the representations of N appearing do not match: $\overline{\alpha}\beta\neq\alpha\beta$ because $\overline{\alpha}\neq\alpha$, and $\alpha\overline{\beta}\neq\alpha\beta$ because $\beta\neq\overline{\beta}$.

Thus, there is a diagram of maps respecting G

$$\begin{array}{ccc} I_\alpha\times I_\beta & \longrightarrow & I_{\alpha\beta} \\ \downarrow \approx & & \\ I_{\overline{\alpha}}\times I_\beta & \longrightarrow & I_{\overline{\alpha}\beta} \end{array} \qquad\qquad (\text{but } I_{\alpha\beta}\not\approx I_{\overline{\alpha}\beta})$$

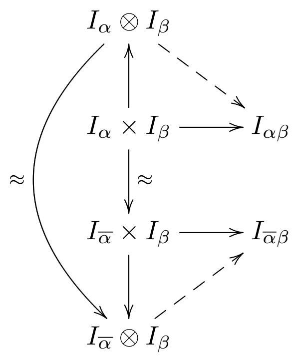
[10] This is the standard model of the principal series, and we adhere to this notation in the sequel.

How can $I_\alpha \times I_\beta$ map to both $I_{\alpha\beta}$ and $I_{\overline{\alpha}\beta}$, while respecting the G-action? This becomes clearer if the $\mathbb{C}$-bilinear multiplication maps are replaced by $\mathbb{C}$-linear maps from *tensor products*, which will respect the G-action. [11] Recall the defining property of a tensor product $V \otimes_{\mathbb{C}} W$ of G-representations V, W: in brief, they convert $\mathbb{C}$-bilinear maps to $\mathbb{C}$-linear. More precisely, given G-representations V, W, there is a $\mathbb{C}$-bilinear $b : V \times W \to V \otimes_{\mathbb{C}} W$ (respecting G) such that for every *bilinear* $V \times W \to X$ (respecting G) there is a unique *linear* $V \otimes_{\mathbb{C}} W \to X$ of G-representations fitting into a commutative diagram [12]

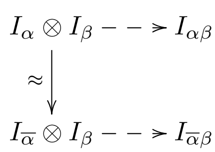


The bilinear map $V \times W \to X$ is said to (uniquely) *factor through* $V \times W \to V \otimes W$. The mapping-property characterization proves that the tensor product is unique up to unique isomorphism, *and* that tensor products of isomorphic G-representations are isomorphic.

Thus, the bilinear multiplication maps factor through linear maps from the corresponding tensor products: sticking the two diagrams together,



More briefly, dropping the bilinear parts of the previous diagram,



The most obvious way one G-representation can surject to two non-isomorphic irreducible G-representations is that it itself is not irreducible, but is the direct sum of the two.

We grant (see the appendix) that computing tensor products interacts well with forgetting group representation structure. Compute the tensor product as N-representations, using distributivity of tensor products over direct sums, and write simply α, rather than $\mathbb{C} \cdot \alpha$, for the one-dimensional representation space on which N acts by α:

$$I_\alpha \otimes I_\beta \text{ (as } N\text{-representation)} \;\approx\; (\alpha \oplus \overline{\alpha}) \otimes (\beta \oplus \overline{\beta}) \;\approx\; \alpha\beta \oplus \alpha\overline{\beta} \oplus \overline{\alpha}\beta \oplus \overline{\alpha}\overline{\beta} \;\approx\; (\alpha\beta \oplus \overline{\alpha}\overline{\beta}) \oplus (\overline{\alpha}\beta \oplus \alpha\overline{\beta})$$

[11] Emphatically, $A \times B \not\approx A \oplus B$. The product $A \times B$ is the cartesian product, not itself naturally a vector space, from which *bilinear* maps make sense. The sum $A \oplus B$ is a vector space in its own right, admitting linear maps to and from it.

[12] Recall that a diagram of maps is *commutative* when the same result is obtained no matter what route is followed. Thus, in the diagram for tensor products, mapping an element of $V \times W$ directly to X gives the same result as mapping from $V \times W$ to $V \otimes W$ and then to X.

Luckily, in the typical situation that $\alpha\beta \neq \overline{\alpha}\overline{\beta}$ and $\overline{\alpha}\beta \neq \alpha\overline{\beta}$, up to isomorphism there is only one G-representation consisting of those four one-dimensional representations of N, namely

$$I_{\alpha\beta} \oplus I_{\overline{\alpha}\beta} \text{ (as } N\text{-representation)} \;\approx\; (\alpha\beta \oplus \overline{\alpha}\overline{\beta}) \oplus (\overline{\alpha}\beta \oplus \alpha\overline{\beta})$$

Thus, at least in the typical case that $\alpha \neq \overline{\alpha}$, $\beta \neq \overline{\beta}$, $\alpha\beta \neq \overline{\alpha}\overline{\beta}$, and $\overline{\alpha}\beta \neq \alpha\overline{\beta}$,

$$I_\alpha \otimes I_\beta \;\approx\; I_{\alpha\beta} \oplus I_{\overline{\alpha}\beta} \qquad \text{(as } G\text{-representation)}$$

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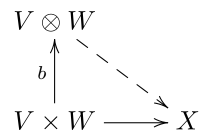
3. *Appendix: tensor products*

Roughly, tensor products of objects of a given sort convert the relevant *bilinear* maps to *linear* maps.

[3.1] Definition More precisely, for modules V, W over a *commutative* ring R, a *tensor product* $V \otimes_R W$ is an R-module, with an *R-bilinear*[13] map $b : V \times W \to V \otimes W$, such that, for every *$R$-bilinear* map $V \times W \to X$ to another R-module X, there is a unique *R-linear* $V \otimes W \to X$ making the following diagram commute:



Say that the map $V \otimes W \to X$ is *induced from*, or *induced by*, the map $V \times W \to X$. For $v \in V$ and $w \in W$, let

$$b(v \times w) \;=\; v \otimes w$$

That is, the symbol $v \otimes w$ is the standard notation for the image of $v \times w$ in $V \otimes_{\mathbb{C}} W$ under the bilinear map b postulated by the definition of tensor product.

[3.2] Tensor products of free modules For complex vector spaces V, W with no additional structure, the tensor product of complex vector spaces $V \otimes_{\mathbb{C}} W$ has dimension equal to the product of the dimensions:

$$\dim_{\mathbb{C}} V \otimes_{\mathbb{C}} W \;=\; \dim_{\mathbb{C}} V \;\cdot\; \dim_{\mathbb{C}} W$$

In fact, this is true for *free modules* V, W over an arbitrary commutative ring R. Specifically, for basis v_i of V and basis w_j of W, the images $v_i \otimes w_j$ form a basis of $V \otimes_R W$. To prove this, we must show that, given any set of supposed images x_{ij} in an R-module X, there is an R-module map $F : V \otimes_R W \to X$ with $F(v_i \otimes w_j) = x_{ij}$ for all indices.

The R-linear map F will be produced from the defining property of the tensor product, by creating an R-bilinear map $\beta : V \times W \to X$ to induce F. Indeed, the bilinear map is the obvious one:

$$\beta\Big(\sum_i a_i v_i \times \sum_j b_j w_j\Big) \;=\; \sum_{ij} a_i b_j \cdot x_{ij} \qquad \text{(with } a_i, b_j \in R\text{)}$$

Since v_i is a basis for V and w_j is a basis for W, trying to define β in this way does produce a well-defined map. The induced map F has the property that

$$F(v_i \otimes w_j) \;=\; \beta(v_i \times w_j) \;=\; x_{ij}$$

---

[13] The notion of *bilinear map* for modules over a *commutative* ring is straightforward: a map $\beta : V \times W \to X$ for R-modules V, W, X is R-linear in each argument separately. That is, $\beta(rv, w) = r\beta(v, w) = \beta(v, rw)$, $\beta(v + v', w) = \beta(v, w) + \beta(v', w)$, and $\beta(v, w + w') = \beta(v, w) + \beta(v, w')$, for $v, v' \in V$, $w, w' \in W$, and $r \in R$. For non-commutative rings there are complications, dealt with a little further below.

This proves that the tensor product is free, on the basis formed from the bases of V and W.

[3.3] Tensor products of representations For two complex representations V, W of a finite group G, the tensor product representation of G is the complex vector space tensor product $V \otimes_{\mathbb{C}} W$ with the action

$$g \cdot (v \otimes w) \;=\; gv \otimes gw$$

If we accept this definition, then the underlying complex vector space does not depend on the group acting, nor whether we choose to look at the action of a proper subgroup.

[3.4] Remark: However, this blunt definition of the group action on the tensor product of complex vector spaces does not explain the group action in categorical terms. The following further discussions briefly establish the larger context.

[3.5] Group rings We want to redescribe G-representation spaces for finite G as modules over a suitable ring, to reduce questions about group representations to questions about modules over (probably non-commutative) rings.

The *group ring* or *group algebra* of a finite group G over the complex numbers is the collection $L^2(G)$ of complex-valued functions on G. The action of $L^2(G)$ on a G-representation V is simply the *averaged* action

$$f \cdot v \;=\; \sum_{g\in G} f(g)\, g \cdot v \qquad\qquad (\text{for } f \in L^2(G) \text{ and } v \in V)$$

The *multiplication* $*$ in $L^2(G)$ appropriate to this action is *not* the pointwise multiplication of functions. The requirement of associativity

$$f \cdot (F \cdot v) \;=\; (f * F) \cdot v \qquad\qquad (\text{for } f, F \in L^2(G) \text{ and } v \in V)$$

determines the multiplication:

$$\sum_{h\in G} f(h)\, h \cdot \big(\sum_{g\in G} F(g)\, g \cdot v\big) \;=\; \sum_{g,h} f(h)F(g)h \cdot (g \cdot v) \;=\; \sum_{g,h} f(hg^{-1})\, F(g)h \cdot v \qquad\qquad (\text{replacing } h \text{ by } hg^{-1})$$

This is

$$\sum_{h} \big(\sum_{g} f(hg^{-1})\, F(g)\big)\, h \cdot v$$

Thus, the multiplication $*$ in the group algebra *must be* the *convolution* product

$$(f * F)(h) \;=\; \sum_{g} f(hg^{-1})\, F(g)$$

Conversely, given a module V over the group algebra $L^2(G)$, we obtain a group representation by

$$g \cdot v \;=\; \delta_g \cdot v \qquad\qquad (\text{with } \delta_g \in L^2(G) \text{ Dirac delta at } g)$$

Rings arising in such a fashion[14] have special features. For example, there is the anti-isomorphism (reversing the order of convolution multiplication) $\iota : L^2(G) \to L^2(G)$ by

$$f^\iota(g) \;=\; f(g^{-1})$$

[14] Group rings provide one of the examples motivating the definition of *Hopf algebra*, which we mention without further discussion.

A direct computation verifies the order-reversing property

$$(f * F)^\iota \;=\; F^\iota * f^\iota$$

That is, this anti-isomorphism ι gives an *isomorphism* of $L^2(G)$ to its *opposite ring*. There is also the *co-diagonal map*, a ring homomorphism

$$\nabla \;:\; L^2(G) \longrightarrow L^2(G) \otimes_{\mathbb{C}} L^2(G)^{\rm op} \qquad \text{by} \qquad \nabla f \;=\; f \otimes f^\iota$$

A *left* $L^2(G)$-module W becomes a *right* $L^2(G)$-module by defining

$$w \cdot f \;=\; f^\iota \cdot w$$

[3.6] Tensor products for non-commutative rings One way to talk about multilinear algebra over non-commutative rings is in terms of *bi-modules*: for two rings R, S, an R, S-*bimodule* V is a *left* R-module and *right* S-module. That is, the notation is that $r \times v \to rv$ for $r \in R$ and $v \in V$, while $s \times v \to vs$ for $s \in S$. The associativities are

$$(rr')v \;=\; r(r'v) \qquad v(ss') \;=\; (vs)s' \qquad r(vs) \;=\; (rv)s \qquad (\text{for } v \in V,\ r, r' \in R, \text{ and } s, s' \in S)$$

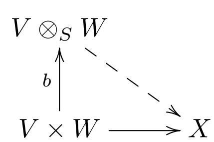
This notation emphasizes that the actions of R and S commute, and also lends itself to discussion of bilinear maps and tensor products for non-commutative [15] rings. There is nothing intrinsic about the use of *left* and *right* here, except as a mnemonic. Indeed, a *right* S-module is indistinguishable from a *left* module over the opposite ring [16] $S^{\rm op}$: let $\theta : S \to S^{\rm op}$ be the identity map on the underlying set, but a ring anti-isomorphism, and then the left $S^{\rm op}$ module structure is given by

$$\theta(s) \cdot v \;=\; v \cdot s \qquad (\text{for } s \in S)$$

For rings R, S, T, for R, S-bimodule V and S, T-bimodule W, a map $\beta : V \times W \to X$ to an R, T-bimodule is R, S, T-bilinear when β is R-linear in V, T-linear in W, and

$$\beta(vs \times w) \;=\; \beta(v \times sw)$$

A *tensor product* over S of such V, W is an R, T-bimodule $V \otimes_S W$ and R, S, T-bilinear map $b : V \times W \to V \otimes_S W$ such that, for every R, S, T-bilinear map $V \times W \to X$, there is a unique R, T-bimodule map $V \otimes_S W \to X$ fitting into the expected commutative diagram



The underlying abelian group of $V \otimes_S W$ does not depend upon the additional R, T structures.

[3.7] Tensor products of representations, revisited Two G-representations V, W are left $L^2(G)$-modules. We can abstract this, to any ring R with an anti-isomorphism $\theta : R \to R^{\rm op}$ of R to its opposite ring, and a co-diagonal ring homomorphism

$$\nabla \;:\; R \longrightarrow R \otimes_{\mathbb{C}} R^{\rm op}$$

Consider V as a $L^2(G), \mathbb{C}$-bimodule. Consider W as a right R-module by $w \cdot \theta(r) = r \cdot w$, and, thus, as a $\mathbb{C}, R^{\rm op}$-bimodule. The tensor product $V \otimes_{\mathbb{C}} W$ is a $R, R^{\rm op}$-bimodule. Via the co-diagonal map $\nabla : R \to R \otimes_{\mathbb{C}} R^{\rm op}$, the tensor product is a R-module. For $R = L^2(G)$, this produces a G-representation.

[3.8] Remark: The last bit of this discussion does not add much to our direct understanding of tensor products of representations, apart from comparison to other (multi-) linear algebra constructs.

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[15] For commutative rings R, an R-module V becomes an R, R-bimodule in the present way of speaking.

[16] As usual, the *opposite ring* of a ring has the same underlying additive group, but the order of multiplication is reversed.