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03. Generalities on representations of finite groups

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A *representation* of a finite group G on a finite-dimensional complex vector space V is a group homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{C}} V$ of G to the $\mathbb{C}$-linear automorphisms of V.

The vector space is completely specified by ρ, so, often, ρ denotes both the *map* from G and the *vectorspace* on which ρ makes G act.

For further notational economy, instead of writing $\rho(g)(v)$, we may write $g \cdot v$ or gv, when context permits.

A *G-morphism* or *G-homomorphism* or *G-map* or *G-intertwining operator*

$$\varphi \ : \ (\sigma, V) \longrightarrow (\tau, W)$$

from one G-representation (σ, V) to another (τ, W) is, as expected, a vector-space map $\varphi : V \to W$ which *commutes with* or *respects* the action of G in the natural sense:

$$\varphi \circ \sigma(g) \ = \ \tau(g) \circ \varphi \qquad \text{(for all } g \in G)$$

Vector-wise, using the lighter notation, the requirement is

$$\varphi(g \cdot v) \ = \ g \cdot \varphi(v) \qquad \text{(for all } v \in V)$$

The G-intertwining operators from (V, σ) to (W, τ) are denoted $\mathrm{Hom}_G(\sigma, \tau)$.

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1. Subrepresentations, complete reducibility, unitarization

A *subrepresentation* of a representation (ρ, V) of G is a $\mathbb{C}$-subspace W of V which is *G-stable* in the sense that, for all $g \in G$ and $W \in W$, $\rho(g)(W) \in W$. With $\rho' : G \to \mathrm{Aut}_{\mathbb{C}} W$ the restriction of $\rho(G)$ to W, (ρ', W) is a representation of G in its own right.

The *direct sum* representation $(\sigma, V) \oplus (\tau, W)$, or simply $\sigma \oplus \tau$, has representation space the direct sum $V \oplus W$ of the two Hilbert spaces, with the natural action

$$(\sigma \oplus \tau)(g)(v \oplus w) \ = \ \sigma(g)v \oplus \tau(g)w$$

That is, more economically,

$$g \cdot (v \oplus w) \ = \ gv \oplus gw$$

A representation (ρ, V) of G is *irreducible* if there is no G-stable subspace of V other than $\{0\}$ and V itself.

[1.1] Remark: A first important idealized goal of representation theory is to *classify* or *parametrize* irreducibles of given G in a useful way.

[1.2] Remark: For G abelian, the irreducibles are exactly given by the group homomorphisms $\chi : G \to \mathbb{C}^\times = \mathrm{Aut}_\mathbb{C}\mathbb{C}$, and are all one-dimensional. This follows from the spectral theorem for finite groups of mutually commuting operators.

A representation (ρ, V) of G is *completely reducible* when it is isomorphic to a direct sum of irreducible representations, that is, when there are irreducibles (ρ_i, V_i) such that

$$\rho \;\approx\; \bigoplus_i \rho_i$$

[1.3] Remark: For finite a abelian group G acting linearly on a finite-dimensional complex vector space, it is an exercise in linear algebra to prove that there is a basis for V of simultaneous eigenvectors for all operators coming from G. This is a decomposition into irreducibles.

[1.4] Remark: A second idealized goal of representation theory is to *use* a classification of irreducibles of given G to usefully describe the irreducibles ρ_i appearing in a decomposition $\rho = \bigoplus_i \rho_i$ of naturally-occurring representations ρ of G, to analyze ρ.

The latter goal, of describing irreducible summands of larger representations, presumes that representations *are* reliably direct sums of irreducibles. This holds for finite-dimensional complex representations of finite groups:

[1.5] Theorem: Every finite-dimensional complex representation of a finite group is *completely reducible.*

Proof: The argument uses the notion of *unitary representation*, and *unitarization* of a given representation. A $\mathbb{C}$-linear map $T : V \to V$ on a finite-dimensional complex Hilbert space V with inner product $\langle,\rangle$ is *unitary* when

$$\langle Tv, Tw\rangle \;=\; \langle v, w\rangle \qquad \text{(for all } v, w \in V)$$

A product of two unitary operators is unitary, as is the inverse of a unitary operator, so the unitary operators on V form a *group.* A representation ρ of G on V is *unitary* when $\rho(g)$ is a unitary operator on V for every $g \in G$.

[1.6] Claim: A representation ρ of G on a complex vector space V is *unitarizable*, meaning that there is a hermitian inner product $\langle,\rangle$ on V so that ρ is *unitary.*

Proof: Take *any* hermitian inner product $\langle,\rangle_o$ on V, and create the invariant $\langle,\rangle$ by *averaging* $\langle,\rangle_o$ over G:

$$\langle v, w\rangle \;=\; \frac{1}{\#G}\sum_{g\in G}\langle gv, gw\rangle$$

By design, this is G-invariant: for $h \in G$,

$$\langle hv, hw\rangle \;=\; \frac{1}{\#G}\sum_{g\in G}\langle g(hv), g(hw)\rangle_o \;=\; \frac{1}{\#G}\sum_{g\in G}\langle (gh)v, (gh)w\rangle_o \;=\; \frac{1}{\#G}\sum_{g\in G}\langle gv, gw\rangle_o$$

by replacing g by g^{-1} in the sum. The averaged inner product is still hermitian-linear: checking linearity in the first argument, for $a \in \mathbb{C}$,

$$\langle av + v', w\rangle \;=\; \frac{1}{\#G}\sum_{g\in G}\langle g(av+v'), gw\rangle_o \;=\; \frac{1}{\#G}\sum_{g\in G}\Big(a\langle gv, gw\rangle_o + \langle gv', gw\rangle_o\Big)$$

$$= \ a\frac{1}{\#G}\sum_{g\in G}\langle gv, gw\rangle_o + \frac{1}{\#G}\sum_{g\in G}\langle gv', gw\rangle_o \ = \ a\langle v,w\rangle + \langle v',w\rangle$$

Conjugate-linearity in the second argument is similar, as is positive-definiteness. ///

Now we can prove the theorem by induction on $\dim_{\mathbb{C}} V$. If there is *no* proper non-zero G-stable subspace of V, then V is irreducible, by definition. If there *is* a proper G-stable subspace W, then, by induction, W is a direct sum $V = \bigoplus_i W_i$ of irreducibles. If we can find a G-stable complementary subspace W' to W, then the induction hypothesis applies to W' as well, so it is a direct sum $W' = \bigoplus_j W'_j$ of irreducibles W'_j, and

$$V \ = \ W \oplus W' \ = \ \bigoplus_i W_i \oplus \bigoplus_j W'_j$$

expresses V as a direct sum of irreducibles. To find a complementary subspace to W, give V a G-invariant hermitian inner product $\langle,\rangle$. We claim that the orthogonal complement W' of W is G-stable: using the G-invariance, that is, the unitariness of the action of G,

$$\langle gw', w\rangle \ = \ \langle g^{-1}gw', g^{-1}w\rangle \ = \ \langle w', g^{-1}\rangle \ = \ 0 \qquad\qquad (\text{for } w' \in W' \text{ and } w \in W)$$

since $g^{-1}W \in W$. This proves the G-stability of the orthogonal complement. Then the induction argument succeeds. ///

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2. *Dual/contragredient representations*

The *dual space* $V^\vee$ of a complex vectorspace V is the complex vectorspace $\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$ of $\mathbb{C}$-valued linear functionals on V. The complex vectorspace structure is

$$(a\cdot\lambda)(v) \ = \ a(\lambda(v)) \qquad\qquad (\text{for } \lambda \in V^\vee,\ a \in \mathbb{C}, \text{ and } v \in V)$$

The *contragredient* or *dual* representation $(\rho^\vee, V^\vee)$ of G on $V^\vee$ is

$$\rho^\vee(g)(\lambda)(v) \ = \ \lambda(\rho(g)^{-1}v)$$

or, more economically,

$$(g\cdot\lambda)(v) \ \ = \ \ \lambda(g^{-1}\cdot v)$$

The possibly unexpected inverse exactly assures that

$$\rho^\vee(gh) \ = \ \rho^\vee(g)\,\rho^\vee(h) \qquad\qquad (\text{for } g,h \in G)$$

The complex-bilinear pairing $V \times V^\vee \to \mathbb{C}$ by $v \times \lambda \to \lambda(v)$ is sometimes usefully denoted

$$\lambda(v) \ = \ \langle v,\lambda\rangle \qquad\qquad (\text{complex bilinear})$$

When V has a hermitian inner product, the latter is often denoted $\langle,\rangle$ as well, inviting confusion. Further, for V with a hermitian inner product, the Riesz-Fischer theorem gives a *complex-conjugate-linear* isomorphism $V \to V^\vee$, by

$$v \ \longrightarrow \ \Big(w \to \langle w,v\rangle\Big) \qquad\qquad (\text{with hermitian inner product})$$

inviting further confusion. We will have reason to use *hermitian* inner products, such as that on $L^2(G)$, and to prove *complete reducibility* as earlier, but the complex *bilinear* pairing of V and $V^\vee$ is equally important. Context should make clear which is meant.

[2.1] Claim: For ρ an irreducible of G, the dual/contragredient $\rho^\vee$ is irreducible.

Proof: For a G-subrepresentation X of $\rho^\vee$, the simultaneous kernel X' of X in ρ is G-stable, because $\lambda(g\cdot v) = (g^{-1}\lambda)(v)$ for all $\lambda \in X$. Since

$$\dim_{\mathbb{C}} X + \dim_{\mathbb{C}} X' \;=\; \dim_{\mathbb{C}} \rho \;=\; \dim_{\mathbb{C}} \rho^\vee$$

the simultaneous kernel X' is a *proper* subspace of ρ if X is a proper subspace of $\rho^\vee$. ///

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3. *Regular and biregular representations*

Let $L^2(G)$ be the square-integrable complex-valued functions on G using counting measure on G. The *right regular representation* R of G on $L^2(G)$ is defined by

$$R(g)f(h) \;=\; R_g f(h) \;=\; f(hg) \qquad\qquad (\text{for } g,h\in G \text{ and } f \in L^2(G))$$

The *left regular representation* L is similarly defined, by

$$L(g)f(h) \;=\; L_g f(h) \;=\; f(g^{-1}h) \qquad\qquad (\text{for } g,h\in G \text{ and } f \in L^2(G))$$

The inverse in the formula is necessary to have $L(gg') = L(g)L(g')$ for non-abelian groups, as in the definition of contragredient representation.

The *biregular representation* of $G\times G$ on $L^2(G)$ is

$$\rho_{\mathrm{bi}}(g\times g')f(h) \;=\; f(g^{-1}hg')$$

[3.1] Claim: The right regular, left regular, and biregular representations of G on $L^2(G)$ are *unitary*.

Proof: For the right regular representation, with $f, F \in L^2(G)$, and $g\in G$, using the definition and changing variables,

$$\langle R_g f, R_g F\rangle \;=\; \sum_{x\in G} f(xg)\,\overline{F(xg)} \;=\; \sum_{x\in G} f(x)\,\overline{F(x)} \;=\; \langle f, F\rangle$$

proving the unitariness. ///

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4. *Schur's Lemma*

[4.1] Theorem: For an irreducible ρ, V of G, a $\mathbb{C}$-linear map $T : V \to V$ commuting with all operators $\rho(g)$ for $g\in G$ is a *scalar*.

Proof: The kernel of such T is G-stable:

$$T(g\cdot v) \;=\; g\cdot Tv \;=\; g\cdot 0 \;=\; 0 \qquad\qquad (\text{for } v\in \ker T)$$

Since ρ is irreducible, $\ker T$ is either $\{0\}$ or V itself. By dimension counting, T is either the 0-map or is a bijection. That is, the ring A of such T contains $\mathbb{C}$, and is a *division algebra*. It is finite-dimensional over $\mathbb{C}$, since V is finite-dimensional. Thus, any $T\in A$ generates a finite algebraic extension of $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, T is scalar. ///

5. *Central characters of irreducibles*

[5.1] Corollary: *(of Schur's lemma)* The center of G acts by *scalars* on an irreducible ρ, V of G. ///

[5.2] Remark: The restriction of ρ to the center of G is the *central character* of ρ, sometimes denoted ω_ρ.

6. *Tensor products of representations*

Given representations σ, V and τ, W of G and H, the (*external*) *tensor product* representation $\sigma\otimes\tau$ of $G\times H$ has representation space $V\otimes_{\mathbb{C}} W$, with action

$$(\sigma\otimes\tau)(g\times h)(v\otimes w) \;=\; \sigma(g)(v)\otimes\tau(h)(w)$$

The (*internal*) *tensor product* of representations σ, V and τ, W of G is defined the same way, but restricting the group action to the diagonal copy G^Δ of G inside $G\times G$. That is,

$$(\sigma\otimes\tau)(g)(v\otimes w) \;=\; \sigma(g)(v)\otimes\tau(g)(w)$$

[6.1] Remark: External and internal tensor products are distinguished by context. Some writers use a *square* tensor symbol for external tensor products, but this is not universal.

[6.2] Theorem: For irreducibles σ, V and τ, W of G and H, the external tensor product $\sigma\otimes\tau$ is an irreducible of $G\times H$.

Proof: We can put a $G\times H$-invariant hermitian inner product $\langle,\rangle$ on $\sigma\otimes\tau$, by averaging. That is, the action of $G\times H$ is by *unitary* operators.

Then, as in the proof of complete reducibility, the orthogonal complement $X^\perp$ of a $G\times H$-stable subspace X is again $G\times H$-stable. Therefore, both the orthogonal projection to X and the orthogonal projection to $X^\perp$ are $G\times H$ maps. They are not scalars. Thus, complementing Schur's lemma, a *reducible* unitary representation of $G\times H$ would have non-scalar $G\times H$ endomorphisms.

To prove $\sigma\otimes\tau$ is irreducible, prove that a $G\times H$-endomorphism T of $\sigma\otimes\tau$ is scalar. For $w\in W$ and $\mu\in W^\vee$ the map $\varphi_{w,\mu}: V\to V\otimes\mathbb{C}\approx V$ defined by

$$v \longrightarrow v\otimes w \longrightarrow T(v\otimes w) \longrightarrow (1\otimes\mu)\Big(T(v\otimes w)\Big)$$

is a G-map. By Schur's lemma, since σ is irreducible,

$$\varphi_{w,\mu}(v) \;=\; \theta_{w,\mu}\cdot v \qquad\qquad (\text{for scalar } \theta_{w,\mu})$$

The map $W^\vee\to W^\vee$ by $\mu\to(w\to\theta_{w,\mu})$ is an H-morphism, since for $h\in H$ and $0\neq v\in V$

$$\theta_{w,h\mu}\cdot v \;=\; (1\otimes h\mu)\Big(T(v\otimes w)\Big) \;=\; (1\otimes\mu)(1_V\otimes h^{-1})\Big(T(v\otimes w)\Big)$$

$$=\; (1\otimes\mu)\Big(T(1_V\otimes h^{-1})(v\otimes w)\Big) \;=\; (1\otimes\mu)\Big(T(v\otimes h^{-1}w)\Big) \;=\; \varphi_{h^{-1}w,\mu}(v) \;=\; \theta_{h^{-1}w,\mu}\cdot v$$

Since W is irreducible, $W^\vee$ is irreducible, and by Schur's lemma this map sends $\mu\to c\mu$ for some $c\in\mathbb{C}$. That is, there is c such that

$$(w\to\theta_{w,\mu}) \;=\; c\cdot(w\to\mu(w))$$

or $\theta_{w,\mu} = c \cdot \mu(w)$. For $\lambda \in V^\vee$ and $\mu \in W^\vee$

$$(\lambda \otimes \mu)\Big(T(v \otimes w)\Big) \;=\; \lambda(\theta_{w,\mu} \cdot v) \;=\; \lambda(v) \cdot c \cdot \mu(w) \;=\; c \cdot \langle v \otimes w, \lambda \otimes \mu\rangle$$

Thus, T acts by the scalar c, and the tensor product is irreducible. ///

[6.3] Remark: Ideas from the above argument are re-used in Schur orthogonality and Schur inner-product formulas.

[6.4] Theorem: For finite G, H, any irreducible ρ of $G \times H$ is isomorphic to $\sigma \otimes \tau$ for irreducibles σ, τ of G and H.

Proof: By complete reducibility, ρ as a G-representation (properly denoted $\mathrm{Res}_G^{G\times H}\rho$) contains an irreducible σ of G. The G-homs $\sigma \to \rho$ are the G-invariant $\mathbb{C}$-linear maps from $\sigma \to \rho$: identifying $\mathrm{Hom}_{\mathbb{C}}(\sigma, \rho)$ with $\sigma^\vee \otimes \rho$,

$$\mathrm{Hom}_G(\sigma, \rho) \;\approx\; (\sigma^\vee \otimes \rho)^G \qquad\qquad \text{(isomorphism of } \mathbb{C}\text{-vectorspaces)}$$

For $v \in \sigma$, $\lambda \in \sigma^\vee$, and $w \in \rho$, the latter action of G is

$$(g \cdot (\lambda \otimes w))(v) \;=\; (g\lambda \otimes gw) \;=\; (g\lambda)(v) \cdot gw \;=\; \lambda(g^{-1}v) \cdot gw$$

The corresponding action on $\mathrm{Hom}_G(\sigma, \rho)$ is then

$$(g \cdot \varphi)(v) \;=\; g \cdot (\varphi(g^{-1}v))$$

The $\mathbb{C}$-vectorspace $\mathrm{Hom}_G(\sigma, \rho)$ is an H-representation, by

$$(h \cdot \varphi)(v) \;=\; h \cdot (\varphi(v)) \qquad\qquad (\text{for } v \in \sigma \text{ and } h \in H)$$

Let τ be an irreducible subrepresentation of $\mathrm{Hom}_G(\sigma, \rho)$. From earlier, $\sigma \otimes \tau$ is an irreducible $G \times H$-representation. The map

$$T : \sigma \otimes \tau \longrightarrow \rho \qquad\qquad \text{by} \qquad\qquad T(v \otimes \varphi) \;=\; \varphi(v)$$

is a $G \times H$-homomorphism. Since ρ is irreducible, and T is not the zero map, it is a surjection. Since $\sigma \otimes \tau$ is irreducible, T is injective. ///

[6.5] Remark: The argument also shows that σ and τ are uniquely determined, up to isomorphism.

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7. *Matrix coefficient functions*

For ρ, V a representation of G, the *matrix coefficient function* attached to $v \in V$ and $\lambda \in V^\vee$ is

$$c_{v,\lambda}(g) \;=\; c^\rho_{v,\lambda}(g) \;=\; \lambda(\rho(g)(v)) \qquad\qquad (\text{for } g \in G,\ v \in V,\ \lambda \in V^\vee)$$

[7.1] Claim: The map $v \otimes \lambda \to c^\rho_{v,\lambda}$ is a $G \times G$-map from $\rho \otimes \rho^\vee$ to the biregular representation on $L^2(G)$.

Proof: The biregular representation's behavior is

$$c_{v,\lambda}(y^{-1}gx) \;=\; \lambda(y^{-1}gx)v \;=\; (y \cdot \lambda)(g(x \cdot v) \;=\; c_{xv,y\lambda}(g)$$

That is, with L the left regular representation of G and R the right regular representation of G on functions on G, we have $L_y R_x c_{v,\lambda} = c_{xv,y\lambda}$. ///

8. *Schur orthogonality*

[8.1] Theorem: Matrix coefficient functions attached to non-isomorphic representations are mutually orthogonal in $L^2(G)$. That is, for σ, V and τ, W non-isomorphic representations of G, for all $v \in V$, $\lambda \in V^\vee$, $w \in W$, and $\mu \in W^\vee$,

$$\langle c^\sigma_{v,\lambda}, c^\tau_{w,\mu}\rangle_{L^2(G)} \;=\; \sum_{g\in G} c^\sigma_{v,\lambda}(g)\,\overline{c^\tau_{w,\mu}(g)} \;=\; 0$$

Proof: Give σ, V a G-invariant inner product. For fixed $\lambda \in V^\vee$, we have a G-map $S : V \to L^2(G)$ by $Sv = c^\sigma_{v,\lambda}$. Similarly, for fixed $\mu \in W^\vee$ we have a G-map $T : W \to L^2(G)$ by $Tw = c^\tau_{w,\mu}$. Then

$$\langle c^\sigma_{v,\lambda}, c^\tau_{w,\mu}\rangle_{L^2(G)} \;=\; \langle Sv, Tw\rangle_{L^2(G)} \;=\; \langle v, (S^*\circ T)w\rangle_V \qquad \text{(with adjoint-map } S^* : L^2(G)\to V)$$

The composition $S^* \circ T$ maps $W \to V$, and commutes with the action of G. Thus, if $S^*\circ T$ were not the zero map it would be a G-isomorphism. But $\sigma \not\approx \tau$, so $S^* \circ T = 0$, and the inner product is 0. ///

[8.2] Remark: A little later, we will give the Schur inner product formula for matrix coefficient functions in the case $\sigma \approx \tau$.

9. *Representations of $C^o_c(G)$, convolutions*

Functions $\varphi \in C^o_c(G)$ act on representations (ρ, V) of G by an *averaged* version of the action:

$$\varphi\cdot v \;=\; \rho(\varphi)(v) \;=\; \sum_{g\in G}\varphi(g)\cdot gv \;=\; \sum_{g\in G}\varphi(g)\cdot\rho(g)(v) \qquad (v\in V,\ \varphi\in C^o_c(G))$$

Since finite groups are *discrete*, the Dirac δ-functions

$$\delta_{x_o}(g) \;=\; \begin{cases} 1 & (\text{for } g = x_o)\\ 0 & (\text{for } g \neq x_o)\end{cases}$$

at $x_o \in G$ are in $C^o_c(G)$. The averaged action of such a function is not really averaged:

$$\rho(\delta_{x_o})v \;=\; \sum_{g\in G}\delta_{x_o}(g)\cdot gv \;=\; 1\cdot x_o v \;=\; x_o v$$

The compactly-supported continuous complex-valued functions $C^o_c(G)$ on G have a uniquely-determined *convolution* product $\varphi\times \longrightarrow \varphi * \psi$ characterized by

$$\rho(\varphi*\psi) \;=\; \rho(\varphi)\circ\rho(\psi) \qquad (\text{for every } \rho)$$

A formula for the convolution follows from this defining property: for fixed $v \in V$,

$$(\rho(\varphi)\circ\rho(\psi))v \;=\; \rho(\varphi)\,(\rho(\psi)v) \;=\; \sum_{x\in G}\varphi(x)\,\rho(x)\left(\sum_{g\in G}\psi(g)\,\rho(g)v\right)$$

$$= \sum_{g\in G}\sum_{x\in G}\varphi(x)\,\psi(g)\,\rho(xg)v \;=\; \sum_{x\in G}\sum_{g\in G}\varphi(xg^{-1})\,\psi(g)\,\rho(x)v$$

by reversing the order of summation and replacing x by xg^{-1}. This is

$$\sum_{x\in G}\Big(\sum_{g\in G}\varphi(xg^{-1})\,\psi(g)\Big)\,\rho(x)v$$

Thus, the inner integral in the latter expression is the convolution (acting on v), that is,

$$(\varphi*\psi)(x) \;=\; \sum_{g\in G}\varphi(xg^{-1})\,\psi(g)$$

giving the desired

$$\rho(\varphi)\circ\rho(\psi) \;=\; \rho(\varphi*\psi) \;=\; \rho(\varphi)\circ\rho(\psi) \qquad\qquad \text{(for every } \rho)$$

Unsurprisingly, convolution is *associative*, as a consequence of its characterization: for f,φ,ψ in $C_c^o(G)$,

$$(f*(\varphi*\psi))(g) \;=\; \sum_{x\in G} f(gx^{-1})(\varphi*\psi)(x) \;=\; \sum_{x\in G}\sum_{y\in G} f(gx^{-1})\,\varphi(xy^{-1})\,\psi(y)$$

$$=\; \sum_{y\in G}\sum_{x\in G} f(gy^{-1}x^{-1})\,\varphi(x)\,\psi(y)$$

by changing the order of summation and replacing x by xy. Then this is

$$\sum_{y\in G}(f*\varphi)(gy^{-1})\,\psi(y) \;=\; ((f*\varphi)*\psi)(g)$$

as asserted.

The Dirac δ at $1\in G$ is the unit in $C_c^o(G)$ with convolution.

[9.1] Proposition: For unitary ρ, the adjoint operator to $\rho(\varphi)$ is $\rho(\varphi^*)$ with

$$\varphi^*(g) \;=\; \overline{\varphi(g^{-1})}$$

Proof: Computing directly,

$$\langle\rho(\varphi)v,w\rangle \;=\; \sum_{g\in G}\varphi(g)\,\langle\rho(g)v,w\rangle \;=\; \sum_{g\in G}\varphi(g)\,\langle v,\rho(g)^*w\rangle \;=\; \sum_{g\in G}\varphi(g)\,\langle v,\rho(g^{-1})w\rangle$$

by unitariness of ρ. Replacing g by g^{-1}, this is

$$\sum_{g\in G}\varphi(g^{-1})\,\langle v,\rho(g)w\rangle \;=\; \sum_{g\in G}\varphi(g^{-1})\,\langle v,\rho(g)w\rangle \;=\; \Big\langle v,\sum_{g\in G}\bar\varphi(g^{-1})\,\rho(g)w\Big\rangle \;=\; \langle v,\rho(\varphi^*)w\rangle$$

as claimed, where the complex conjugate appears because the inner product is conjugate-linear in its second argument. ///

10. *G-representations versus $C_c^o(G)$-modules*

The category of group representations of G and G-maps among them is *the same* as the category of $C_c^o(G)$-modules and $C_c^o(G)$-modules homomorphisms: for representations (ρ, V) and (τ, W) of G,

• The collection of G-stable subspaces of V is identical to the collection of $C_c^o(G)$-stable subspaces of V.

• A linear map $T : V \to W$ is a G-homomorphism if and only if it is an $C_c^o(G)$-module homomorphism.

• The representation (ρ, V) is G-irreducible if and only if it is $C_c^o(G)$-irreducible.

That is, these two categories are *equivalent*, in the sense that the objects are in bijection, and the morphisms are in natural bijection.

11. *Decomposition of biregular representation on $L^2(G)$*

For irreducible ρ of G, formation of matrix coefficient functions

$$\rho \otimes \rho^\vee \to L^2(G) \qquad \text{by} \qquad v \otimes \lambda \to c_{v,\lambda}$$

is a $G \times G$ map. The Schur orthogonality relations demonstrate the *mutual orthogonality* of the images for non-isomorphic irreducibles ρ.

For ρ an irreducible unitary representation of G, let $L^2(G)^\rho$ denote the ρ-*isotypic component* inside $L^2(G)$ under the right regular representation R, that is, the sum of all copies of ρ in $L^2(G)$ with right regular representation of G. Certainly $L^2(G)^\rho \supset \rho \otimes \rho^\vee$.

[11.1] **Theorem:** The isotypic component $L^2(G)^\rho$ under the right regular representation is *stable* under the *left* regular representation L, and as $G \times G$-representation with the biregular representation,

$$L^2(G)^\rho \approx \rho \otimes \rho^\vee$$

Therefore, as a $G \times G$ representation,

$$L^2(G) \;\approx\; \bigoplus_\rho \rho \otimes \rho^\vee \qquad \text{(sum over } G\text{-irreducibles } \rho)$$

Proof: Again Schur orthogonality gives directness of the sum, in

$$L^2(G) \;\supset\; \bigoplus_\rho \rho \otimes \rho^\vee \qquad \text{(sum over } G\text{-irreducibles } \rho)$$

It remains to be shown that there is *nothing else* in $L^2(G)$. By complete reducibility, it suffices to show that the ρ-isotype under the right regular representation is no larger than the image of $\rho \otimes \rho^\vee$.

For f in a copy of ρ inside $L^2(G)$ *orthogonal* to all coefficient functions $c^\rho_{v,\lambda}$ coming from $\rho \otimes \rho^\vee$, by the stability of matrix coefficient functions under the left regular representation and the unitariness of the biregular representation, all left translates $L_g f$ are still orthogonal to all these coefficient functions.

With real-valued $\varphi \in C_c^o(G)$, let $\varphi^\vee(g) \;=\; \varphi(g^{-1})$. By the equivalence of categories of G-representations and $C_c^o(G)$-modules, there is φ such that

$$0 \neq \varphi^\vee \cdot f \;=\; \Big(g \longrightarrow \sum_{h\in G} f(h^{-1}g)\,\varphi^\vee(h)\Big) \;=\; \Big(g \longrightarrow \sum_{h\in G} f(hg)\,\varphi(h)\Big)$$

With η the orthogonal projection of φ to the ρ-isotype,

$$\varphi^\vee f \;=\; c_{f,\varphi} \;=\; c_{f,\eta}$$

and

$$\langle \rho(\varphi^\vee) f, c^\rho_{v,\lambda}\rangle \;=\; \langle c^\rho_{f,\eta}, c^\rho_{v,\lambda}\rangle \;=$$

In particular, taking $v = f$ and $\lambda = \eta$ gives a contradiction: there is no non-trivial f in the ρ-isotype orthogonal to all the matrix coefficient functions. That is, $\rho \otimes \rho^\vee$ is the whole ρ-isotype. ///
