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03b. More generalities on representations of finite groups

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1. *Characters as projectors*

The *character* χ_ρ of a representation ρ of a finite group G is

$$\chi_\rho(g) \;=\; \mathrm{trace}\Big(\rho(g)\Big) \qquad\qquad (\text{for } g\in G)$$

As a complex-valued function on G, χ_ρ acts on any representation σ of G, by

$$\chi_\rho\cdot v \;=\; \sum_{g\in G}\chi_\rho(g)\cdot\sigma(g)v \qquad\qquad (\text{for } v\in\sigma)$$

[1.1] Theorem: For any representation σ of G, the character $\chi_{\rho^\vee}$ of $\rho^\vee$ essentially acts on σ as the *projector* to the ρ-isotype σ^ρ of σ:

$$\frac{\dim\rho}{\#G}\cdot\sigma(\chi_{\rho^\vee}) \;=\; \text{projector to } \sigma^\rho$$

In particular, $\rho(\chi_{\rho^\vee})$ is the scalar $\#G/\dim\rho$ on ρ itself, and on *other* irreducibles is the scalar 0. Also, as a by-product of the proof,

$$\sum_{g\in G}\chi_\rho(g)\cdot\chi_\rho^\vee(g) \;=\; \#G$$

Proof: The conjugation-invariance of *trace* shows that $\sigma(\chi_\rho)$ commutes with all operators $\sigma(g)$:

$$\sigma(g)\circ\sigma(\chi_\rho)\circ\sigma(g)^{-1} \;=\; \sigma(g)\circ\sum_{h\in G}\mathrm{tr}(\rho(h))\cdot\sigma(h)\circ\sigma(g)^{-1} \;=\; \sum_{h\in G}\mathrm{tr}(\rho(h))\cdot\sigma(ghg^{-1})$$

$$=\; \sum_{h\in G}\mathrm{tr}(\rho(g^{-1}hg))\cdot\sigma(h) \;=\; \sum_{h\in G}\mathrm{tr}(\rho(h)\cdot\sigma(h) \;=\; \sigma(\chi_\rho)$$

Thus, for σ irreducible, by Schur's lemma $\sigma(\chi_\rho)$ is *scalar*. For finite-dimensional vector spaces V, the natural map

$$V\otimes_{\mathbb{C}}V^\vee \;\longrightarrow\; \mathrm{End}_{\mathbb{C}}V \qquad \text{by} \qquad (v\otimes\lambda)(w)=\lambda(w)\cdot v \qquad (\text{for } v,w\in V \text{ and } \lambda\in V^\vee)$$

is an *isomorphism*, by dimension-counting. For V a G-representation, this map is a G-isomorphism:

$$\Big(g\cdot(v\otimes\lambda)\Big)(w) \;=\; (gv\otimes g\lambda)(w) \;=\; (g\lambda)(w)\cdot(gv) \;=\; \lambda(g^{-1}w)\cdot(gv) \;=\; \Big(g\circ(v\otimes\lambda)\circ g^{-1}\Big)(w)$$

where G acts on $\varphi\in\mathrm{End}_{\mathbb{C}}V$ by $\varphi\to g\circ\varphi\circ g^{-1}$. This canonically identifies G-invariants in $\rho\otimes\rho^\vee$, via Schur's lemma:

$$(\rho\otimes\rho^\vee)^G \;=\; \mathrm{Hom}_{\mathbb{C}}(\rho,\rho)^G \;=\; \mathrm{Hom}_G(\rho,\rho) \;=\; \mathbb{C}\cdot 1_\rho \qquad\qquad (1_\rho \text{ the identity map on } \rho)$$

Trace on $\mathrm{End}_{\mathbb{C}}\rho$ is the canonical extension of $v \otimes \lambda \to \lambda(v)$, and $\mathrm{tr}(1_\rho) = \dim_{\mathbb{C}} \rho$.

Given $v \otimes \lambda \in \rho \otimes \rho^\vee$, the normalized averaging [1]

$$v \otimes \lambda \;\longrightarrow\; \frac{1}{\#G} \sum_{g \in G} g \cdot (v \otimes \lambda)$$

produces a G-invariant in $\rho \otimes \rho^\vee$, necessarily a scalar multiple of 1_ρ:

$$\frac{1}{\#G} \sum_{g \in G} g \cdot (v \otimes \lambda) \;=\; C_\rho(v, \lambda) \cdot 1_\rho$$

Taking trace (of endomorphisms of ρ) determines the scalar: using G-invariance of trace,

$$C_\rho(v,\lambda) \cdot \dim \rho \;=\; \mathrm{tr}\Big(C_\rho(v,\lambda) \cdot 1_\rho\Big) \;=\; \mathrm{tr}\Big(\frac{1}{\#G} \sum_{g \in G} g \cdot (v \otimes \lambda)\Big) \;=\; \frac{1}{\#G} \sum_{g \in G} \mathrm{tr}\Big(g \cdot (v \otimes \lambda)\Big)$$

$$=\; \frac{1}{\#G} \sum_{g \in G} \mathrm{tr}(v \otimes \lambda) \;=\; \mathrm{tr}(v \otimes \lambda) \;=\; \lambda(v)$$

Thus,

$$\frac{1}{\#G} \sum_{g \in G} g \cdot (v \otimes \lambda) \;=\; \frac{\lambda(v)}{\dim \rho} \cdot 1_\rho$$

The scalar C_ρ by which $\rho(\chi_{\rho^\vee})$ acts on ρ is determined by taking trace of the endomorphism $\rho(\chi_{\rho^\vee})$:

$$\dim \rho \cdot C_\rho \;=\; \mathrm{tr}\Big(\rho(\chi_{\rho^\vee})\Big) \;=\; \mathrm{tr}\Big(\sum_{g \in G} \chi_{\rho^\vee}(g) \cdot \rho(g)\Big) \;=\; \sum_{g \in G} \chi_{\rho^\vee}(g) \cdot \mathrm{tr}(\rho(g)) \;=\; \sum_{g \in G} \chi_{\rho^\vee}(g) \cdot \chi_\rho(g)$$

For a basis $\{v_i\}$ of ρ and corresponding dual basis $\{\lambda_i\}$ of $\rho^\vee$, and identifying $\rho^{\vee\vee} = \rho$, this is

$$\sum_{g \in G} \sum_i v_i(g\lambda_i) \cdot \sum_j \lambda_j(gv_j) \;=\; \sum_{i,j} (v_i \otimes \lambda_j)\Big(\sum_g g(\lambda_i \otimes v_j)\Big) \;=\; \sum_{i,j} (v_i \otimes \lambda_j)\Big(\#G \cdot \frac{\lambda_i(v_j)}{\dim \rho} \cdot 1_{\rho^\vee}\Big)$$

$$=\; \frac{\#G}{\dim \rho} \sum_i (v_i \otimes \lambda_i)(1_{\rho^\vee}) \;=\; \frac{\#G}{\dim \rho} \cdot 1_\rho(1_{\rho^\vee}) \;=\; \#G$$

That is,

$$\dim \rho \cdot C_\rho \;=\; \dim \rho \cdot (\text{scalar by which } \chi_{\rho^\vee} \text{ acts on } \rho) \;=\; \sum_{g \in G} \chi_{\rho^\vee}(g) \cdot \chi_\rho(g) \;=\; \#G$$

Similarly, Schur orthogonality shows that $\chi_{\rho^\vee}$ acts by 0 on any irreducible other than ρ. ///

[1] This averaging is normalized so that it is *idempotent*, that is, averaging *twice* produces the same result as averaging *once*.

2. *Schur inner product relations*

[2.1] Claim: On *irreducible* ρ, V of G, there is a *unique* G-invariant hermitian inner product, up to positive real scalar multiples.

Proof: *At least one* G-invariant inner product can be created by *averaging* an arbitrary inner product. The resulting $\mathbb{C}$-bilinear map $V \times \overline{V} \to \mathbb{C}$ gives a G-invariant $\mathbb{C}$-linear $V \otimes \overline{V} \to \mathbb{C}$.

Using the Cartan-Eilenberg adjunction, and M^G denoting the G-invariant elements of a module M,

$$\mathrm{Hom}_G(A \otimes B, C) \;=\; \mathrm{Hom}_{\mathbb{C}}(A \otimes B, C)^G \;\approx\; \mathrm{Hom}_{\mathbb{C}}(A, \mathrm{Hom}(B, C))^G \;=\; \mathrm{Hom}_{\mathcal{G}}(A, \mathrm{Hom}(B, C))$$

In particular, with C the trivial G-representation $\mathbb{C}$,

$$\mathrm{Hom}_G(A \otimes B, \mathbb{C}) \;=\; \mathrm{Hom}_{\mathbb{C}}(A \otimes B, \mathbb{C})^G \;\approx\; \mathrm{Hom}_{\mathbb{C}}(A, \mathrm{Hom}(B, \mathbb{C}))^G \;=\; \mathrm{Hom}_G(A, B^\vee)$$

For G-irreducibles A, B, since $B^\vee$ is irreducible, by Schur's lemma $\mathrm{Hom}_G(A, B^\vee)$ is 0 unless $A \approx B^\vee$, in which case this space is one-dimensional. Taking $A = V$ and $B = \overline{V}$ gives the claim. ///

[2.2] Remark: Given irreducible ρ, V of G with essentially unique invariant $\langle , \rangle$, the finite-dimensional version of Riesz-Fischer gives a conjugate-linear isomorphism $\rho^\vee \approx \overline{\rho}$ depending on the inner product. The conjugated space $\overline{\rho}$ inherits the hermitian inner product from ρ, V, from which $\rho^\vee \approx \rho$ inherits an inner product. Changing the invariant inner product on ρ by a scalar changes the isomorphism $\rho^\vee \approx \overline{\rho}$, and changes the inner product on $\rho^\vee$ by the *inverse* scalar. Thus, the assertion in the following theorem is independent of choice of the ambiguous scalar in the inner product.

[2.3] Theorem: *(Schur)* For irreducible ρ of G with G-invariant inner product,

$$\langle c^\rho_{v,\lambda}, c^\rho_{w,\mu} \rangle_{L^2(G)} \;=\; \sum_{g \in G} c^\rho_{v,\lambda}(g)\, \overline{c^\rho_{w,\mu}}(g) \;=\; \frac{\#G}{\dim \rho} \langle v, w \rangle\, \overline{\langle \lambda, \mu \rangle} \qquad \text{(for all } v, w \in V,\ \lambda, \mu \in V^\vee)$$

[2.4] Remark: Some sources normalize the inner product on $L^2(G)$ by dividing by $\#G$, which has the effect of eliminating the $\#G$ in the Schur formula.

Proof: Let $Sv = c^\rho_{v,\lambda}$ and $Tw = c^\rho_{w,\mu}$. Again, $S^* \circ T$ is a G-map $V \to V$, so by Schur's lemma is a scalar, and

$$\langle c_{v,\lambda}, c_{w,\mu} \rangle_{L^2(G)} \;=\; \langle Sv, Tw \rangle_{L^2(G)} \;=\; \langle v, (S^* \circ T) w \rangle_V \;=\; C_{\lambda,\mu} \cdot \langle v, w \rangle_V$$

for a scalar $C_{\lambda,\mu}$. Similarly, complex conjugating and replacing g by g^{-1}, for some scalar $D_{v,w}$

$$\overline{\langle c^\rho_{v,\lambda}, c^\rho_{w,\mu} \rangle} \;=\; \sum_{g \in G} \overline{c^\rho_{v,\lambda}(g)}\, c^\rho_{w,\mu}(g) \;=\; \sum_{g \in G} c^{\rho^\vee}_{\lambda,v}(g)\, \overline{c^{\rho^\vee}_{\mu,w}(g)} \;=\; \overline{D}_{v,w} \cdot \langle \lambda, \mu \rangle$$

Thus,

$$\frac{\overline{\langle \lambda, \mu \rangle}}{C_{\lambda,\mu}} \;=\; \frac{\langle v, w \rangle}{D_{v,w}} \qquad \text{(for all } v, w, \lambda, \mu)$$

Since the left-hand side does not depend on v, w, and the right-hand side does not depend on λ, μ, both sides are a constant C depending only on ρ and on the choices of G-invariant inner products, and

$$C \cdot C_{\lambda,\mu} \;=\; \overline{\langle \lambda, \mu \rangle} \qquad \text{and} \qquad C \cdot D_{v,w} \;=\; \langle v, w \rangle$$

Thus, with a constant $C = C_\rho$ depending only on ρ,

$$\langle c_{v,\lambda},\, c_{w,\mu}\rangle_{L^2(G)} \;=\; \frac{1}{C}\cdot\langle v,w\rangle\cdot\overline{\langle\lambda,\mu\rangle}$$

To evaluate the constant, use the earlier computations on inner products of characters, namely, for orthonormal basis $\{v_i\}$ of ρ and corresponding dual basis $\{\lambda_i\}$ for $\rho^\vee$,

$$\#G \;=\; \langle\chi_\rho,\,\chi_\rho\rangle \;=\; \sum_{ij}\langle c_{v_i,\lambda_i},\, c_{v_j,\lambda_j}\rangle \;=\; \dim\rho\cdot\frac{1}{C}$$

Thus, $C = C_\rho = \dfrac{\dim\rho}{\#G}$. ///

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3. *Traces, characters, central functions*

The *central functions* $L^2_{\rm cen}(G)$ in $L^2(G)$ are the conjugation-invariant functions:

$$L^2_{\rm cen}(G) \;=\; \{f\in L^2(G) : f(h^{-1}gh) = f(g) \text{ for } h,g\in G\}$$

The space of central functions is not generally stable under right or left translation by G, but only under the conjugation action of G

$$\rho_{\rm conj}(h)f(g) \;=\; f(h^{-1}gh)$$

Since trace is invariant under conjugation, every character χ_ρ is a central function.

[3.1] Theorem: The collection of characters χ_ρ of irreducibles ρ is an orthogonal basis for $L^2_{\rm cen}(G)$, with

$$\langle\chi_\rho,\chi_\rho\rangle \;=\; \#G$$

Proof: The inner product of χ_ρ with itself was computed above. The orthogonalities follow from the expression of χ_ρ in terms of matrix coefficient functions, from Schur's inner product relations.

Since

$$L^2(G) \;=\; \bigoplus_{\text{irred }\rho}\rho\otimes\rho^\vee$$

and $\rho\otimes\rho^\vee$ is conjugation-stable, it suffices to show that the *central* functions in $\rho\otimes\rho^*$ are exactly the multiples of χ_ρ. The central functions in $\rho\otimes\rho^*$ are

$$(\rho\otimes\rho^\vee)^G \;\approx\; \mathrm{Hom}_G(\rho,\rho) \;\approx\; \mathbb{C}$$

since ρ is irreducible, by Schur's lemma. That is, the space of central functions in $\rho\otimes\rho^*$ is one-dimensional, so must be just $\mathbb{C}\cdot\chi_\rho$. ///

[3.2] Corollary: With G as above, the characters of mutually non-isomorphic irreducible unitary representations are linearly independent. ///

[3.3] Corollary: Two irreducible unitary representations of G are isomorphic if and only if their characters are equal. ///
