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05. Heisenberg groups over finite fields, Segal-Shale-Weil

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1. Heisenberg groups

Fix a field k not of characteristic 2.

[1.1] Tangible models A small Heisenberg group over k can be modelled by the group of matrices of the form[1]

$$H = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in k \}$$

Larger Heisenberg groups over k are formed by replacing scalar x, y by vectors:

$$H = \{ \begin{pmatrix} 1 & x & z \\ 0 & 1_n & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & x_2 & \dots & x_n & z \\ 0 & 1 & 0 & \dots & 0 & y_1 \\ 0 & 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & \vdots & & \ddots & 0 & y_{n-1} \\ 0 & 0 & \dots & 0 & 1 & y_n \\ 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} : x = (x_1 \ \ \dots \ \ x_n), \ y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \ z \in k \}$$

Multiplication is

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + x' & z + z' + xy' \\ 0 & 1 & y + y' \\ 0 & 0 & 1 \end{pmatrix}$$

where the 1-by-1 product xy' of 1-by-n and n-by-1 matrices is viewed as a scalar. Inverses are

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -x & -z + xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}$$

The center Z is small:

$$Z = \{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in k \}$$

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[1] Strictly speaking, these are *non-reduced* Heisenberg groups. Various quotients by subgroups of the center, *reduced* Heisenberg groups are also technically useful.

Commutators are

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & xy' - x'y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This introduces an *alternating form* $\langle , \rangle$ on k^{2n}:

$$\langle x \oplus y,\, x' \oplus y' \rangle \;=\; xy' - x'y \qquad \text{(with } x, x' \text{ row vectors, } y, y' \text{ columns)}$$

The commutator computation shows that H/Z is *abelian*.

[1.2] Remark: There are obvious abstractions and de-coordinatizations of the above. For example, we can fix a k-vectorspace W, let W^* be its dual, and define a group structure on $W^* \oplus W \oplus k$ by the same pattern of symbols

$$(\lambda, w, z)\,(\lambda', w', z') \;=\; (\lambda + \lambda',\, w + w',\, z + z' + \lambda w') \qquad \text{(with } w, w' \in W,\ \lambda, \lambda' \in W^*, \text{ and } z, z' \in k)$$

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2. *Uniqueness for given non-trivial central character*

The general assertion is that, given a non-trivial central character of a given Heisenberg group, there is a unique irreducible with that central character.

Let k be a finite field with q elements, q odd. The cardinality of the n^{th} Heisenberg group H over k is q^{2n+1}.

On an irreducible complex representation π of H the center Z acts by a character ω, by Schur's Lemma.

Since $H/Z \approx k^{2n}$ is abelian, the irreducibles with *trivial* central character ω are one-dimensional, the q^{2n} characters of the additive group k^{2n}.

The irreducibles π with non-trivial central character ω are of greater interest. The restriction of such π to the (maximal) abelian subgroup

$$A \;=\; \{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H \} \;\approx\; k^n \oplus k \}$$

decomposes as a sum of one-dimensional irreducibles, each of which is ω on Z. The commutation relation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & -xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

shows that when one character ψ of A appears in the restriction of π to A, necessarily

$$\psi\Big(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}^{-1} \Big) = \psi \begin{pmatrix} 1 & x & -xy \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \psi(x) \cdot \omega(-xy)$$

also appears. For non-trivial ω, as $y \in k^n$ varies, $x \to \omega(-xy)$ ranges through all characters on k^n. That is, when π has non-trivial central character, *all* characters of $A \approx k^n \oplus k$ restricting to ω on $Z \approx k$ appear at least once in π restricted to A. There are q^n such characters for fixed ω, so *the dimension of π with any non-trivial central character is at least q^n.*

Existence of an irreducible with given (non-trivial or not) central character ω follows from existence of the induced representation $\mathrm{Ind}_A^H \psi$, where ψ is any character on A that restricts to ω on Z. Since $\#(H/A) = q^n$, the number of characters ψ of A extending ω, each such induced representation is *irreducible*. That is, for non-trivial central character ω and any ψ extending ω to A, $\mathrm{Ind}_A^H \psi$ is irreducible.

The finiteness quickly shows there is *exactly one* irreducible π with given non-trivial central character ω, since the sum of squares of dimensions of irreducibles is the cardinality of the group. From the inequality

$$q^{2n+1} \;=\; |H| \;=\; \sum_{\mathrm{one-dim'l}\ \pi} 1^2 + \sum_{\mathrm{non-trivial}\ \omega} (\dim \pi)^2 \;\geq\; q^{2n} + (q-1)\cdot (q^n)^2 \;=\; q^{2n+1}$$

we find that the inequality is an equality. Thus, the dimension of π with non-trivial central character is exactly q^2, and each character of A that restricts to ω on Z occurs exactly once. That is, *for given non-trivial central character ω, up to isomorphism there is a unique irreducible π with central character ω.*

[2.1] **Remark:** This uniqueness is a trivial analogue of the *assertion* of the Stone-vonNeumann theorem, which makes the corresponding assertion over $\mathbb{R}$. Of course, no dimension-counting argument is available over $\mathbb{R}$.

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3. *Automorphisms of Heisenberg groups*

Given an irreducible π of H, and given an automorphism α of H, the twist $\pi^\alpha : h \to \pi(\alpha h)$ is a representation of H on the same representation space as that of π. For automorphisms α trivial on the center Z of H, the twist π^α has the same central character ω as π. By the uniqueness of the irreducibles with given non-trivial central character ω, for automorphism α trivial on Z, $\pi^\alpha \approx \pi$ as H-representations.

The group of automorphisms of H fixing Z element-wise is large, as becomes visible with a more coordinate-independent description of the group. Although the matrix description gives an immediate sense of how the group behaves, it accidentally makes some misleading distinctions via the x, y coordinates above.

In particular, we show that, for a given field k (not characteristic 2), a $2n$-dimensional k-vectorspace V with a non-degenerate alternating form $\langle,\rangle$ completely specifies a Heisenberg-type group $H = H(V, \langle,\rangle)$, and any linear automorphism of V preserving the alternating form gives an automorphism of H.

First, reconsider a Heisenberg group in coordinates. Because the characteristic is not 2 there is an *exponential* map

$$\exp\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;=\; 1_{n+2} + \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} + \tfrac{1}{2}\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}^2 \;=\; \begin{pmatrix} 1 & x & z+\frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

from the Lie algebra

$$\mathfrak{h} \;=\; \{\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}\}$$

of the Heisenberg group H to the Heisenberg group. The coordinates (x, y, z) on the Lie algebra turn out to be better than the seemingly-natural coordinates on the group H: we will use notation

$$(x,y,z) \;=\; \exp\begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \;=\; \begin{pmatrix} 1 & x & z+\frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

Ignoring the center,

$$(x,y,0)\cdot(x',y',0) \;=\; \begin{pmatrix} 1 & x & \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 & x' & \frac{x'y'}{2} \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} \;=\; \begin{pmatrix} 1 & x+x' & \frac{xy}{2}+\frac{x'y'}{2}+xy' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (x+x', y+y', 0)\cdot(0,0,\frac{xy'-x'y}{2})$$

That is, in Lie algebra/exponential coordinates, letting $v=(x,y,0)$ and $v'=(x',y',0)$,

$$v\cdot v' \;=\; (v+v')\cdot(0,0,\frac{\langle v,v'\rangle}{2})$$

Using the (x,y,z) coordinates on the Lie algebra $\mathfrak{h}$, the Lie bracket is

$$[(x,y,z),\,(x',y'z')] \;=\; \begin{pmatrix}0&x&z\\0&0&y\\0&0&0\end{pmatrix}\begin{pmatrix}0&x'&z'\\0&0&y'\\0&0&0\end{pmatrix} - \begin{pmatrix}0&x'&z'\\0&0&y'\\0&0&0\end{pmatrix}\begin{pmatrix}0&x&z\\0&0&y\\0&0&0\end{pmatrix}$$

$$= \begin{pmatrix}0&0&xy'-x'y\\0&0&0\\0&0&0\end{pmatrix} \;=\; (0,0,xy'-x'y)$$

Abstracting this computation, for given k-vectorspace V with non-degenerate alternating form $\langle,\rangle$, put a Lie algebra[2] structure $\mathfrak{h}$ on $V\oplus k$ by Lie bracket

$$[v\oplus z,\, v'\oplus z'] \;=\; 0\oplus\langle v,v'\rangle$$

In exponential coordinates on H, the exponential map $\mathfrak{h}\to H$ with $H\approx V\oplus k$ is notated

$$\exp(v\oplus z) \;=\; v\oplus z$$

with Lie group structure on H by

$$(v\oplus z)\cdot(v'\oplus z') \;=\; (v+v')\oplus(z+z'+\frac{\langle v,v'\rangle}{2}) \qquad \text{(exponential coordinates in } H)$$

In the Lie algebra/exponential coordinates, any k-linear map $g:V\to V$ preserving the alternating form gives an automorphism τ_g of the Lie algebra $\mathfrak{h}\approx V\oplus k$ and and of the Lie group $H\approx V\oplus k$, by

$$\tau_g(v\oplus z) \;=\; gv\oplus z \qquad \text{(same expression for both } \mathfrak{h} \text{ and } H)$$

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4. *Segal-Shale-Weil/oscillator representations*

The uniqueness of the representation π with fixed non-trivial central character ω of the Heisenberg group $H=V\oplus k$, *almost* gives a representation of the isometry group $Sp(V)=Sp(V,\langle,\rangle)$ of $\langle,\rangle$ on V.

What literally arises is a *projective* representation ρ of $Sp(V)$ on π, meaning that each $\rho(g)$ is ambiguous by a scalar depending on g, as follows. Let π be an irreducible of H on a complex vectorspace X, with non-trivial central character ω. The twist π^g given by $\pi^g(h)(x)=\pi(\tau_g h)(x)$ is a representation of H on the same vectorspace X.

By uniqueness, the twist π^g is H-isomorphic to π.[3] By Schur's lemma, this isomorphism is unique *up to constants*. That is, there is an H-isomorphism $\rho(g):(\pi^g,X)\to(\pi,X)$, unique up to scalar multiples.

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[2] In positive characteristic, generally a Lie algebra needs some further structure to behave properly, but in this simple situation the potential troubles do not appear.

[3] The map $(\pi,X)\to(\pi^g,X)$ by the identity mapping on X is most likely *not* an H-homomorphism.

That is, $\rho(g) : X \to X$ is a (linear) automorphism of X.

For a subgroup $\Theta \subset Sp(V)$, a family of choices $\{\rho(g)\}$ isomorphism $\rho(g) : X \to X$ with the multiplicative compatibility

$$\rho(xy) \;=\; \rho(x) \circ \rho(y) \qquad\qquad (\text{for } x, y \in \Theta)$$

is a *Segal-Shale-Weil/oscillator* representation of Θ.

The only possible adjustment of an initial choice of $\rho(g)$'s, to try to achieve the multiplicative compatibility, is by *constants*. There is no advance assurance that adjustment of constants to achieve multiplicative compatibility is possible. [4]

Over a finite field, we *can* exhibit a compatible family of choices for g in a subgroup of the full isometry group $Sp(V)$. For example, $Sp(V)$ has subgroups of the form $O(Q) \times Sp(V') \subset Sp(V)$ where Q is a non-degenerate quadratic space, V' is a non-degenerate alternating space, and $V \approx Q \otimes V'$. The alternating form on $Q \otimes V'$ is

$$\langle x \otimes v, x' \otimes v' \rangle \;=\; \langle x, x' \rangle_Q \cdot \langle v, v' \rangle_{V'}$$

where $\langle , \rangle_Q$ is the symmetric form on Q and $\langle , \rangle_{V'}$ is the alternating form on V'. The inclusion $O(Q) \times Sp(V') \subset Sp(V)$ is induced by the linear isomorphism $Q \otimes V' = V$.

[4.1] Canonical intertwinings

There is a constructions of a family of copies of π facilitating discussion of twists π^g by $g \in Sp(V)$.

A *Lagrangian* or *maximal totally isotropic* subspace W of V is one on which $\langle w_1, w_2 \rangle = 0$ for all $w_1, w_2 \in W$. For fixed non-trivial central character ω, given a Lagrangian subspace W, extend ω trivially to $W \oplus Z \approx \exp(W \oplus Z)$ by $\omega(\exp W) = \{1\}$ and still denote this extension by ω. Let

$$I_W \;=\; \mathrm{Ind}_{W\oplus Z}^H \,\omega \;=\; \{\mathbb{C}\text{-valued } f \text{ on } H : f(ah) = \omega(a) \cdot f(h) \text{ for } a \in \exp(W \oplus Z)\}$$

By uniqueness, the unique (isomorphism class) of irreducible(s) with central character ω is $\pi \approx I_W$.

A function $f \in I_W$ is determined by its values on any Lagrangian subspace complementary to W. [5]

For a linear map $g : V \to V$ preserving the alternating form, writing $(g \cdot f)(h) = f(\tau_g h)$, the *twist* $\pi^g \approx I_W^g$ is identifiable as

$$\{\tau_g f \text{ for } f \in I_W\} \;=\; \{f^g \text{ on } H : \tau_g f(ah) = f(a^g\, h^g) = \omega(a^g) \cdot \tau_g f(h) \text{ for } a \in \exp(W \oplus Z)\} \;=\; I_{W^g}$$

By uniqueness, for fixed non-trivial ω all these twists are H-isomorphic. Conveniently, there is a natural explicit expression for an intertwining $T_{W\to W'} : I_W \to I_{W'}$ for any two Lagrangian subspaces W, W':

$$\big(T_{W\to W'} f\big)(h) \;=\; \sum_{u \in W'/(W'\cap W)} f(e^u \cdot h) \qquad\qquad (\text{writing } e^u = \exp(u))$$

Such a map is the identity for $W = W'$. By Schur, every such map is either 0 or an H-isomorphism. To see that such an intertwining is non-zero, apply it to the function $\delta_W \in I_W$ given by

$$\delta_W(h) \;=\; \begin{cases} 0 & \text{for } h \notin e^{W+Z} \\ \\ \omega(z) & \text{for } h = e^{w+z} = e^w e^z \text{ with } w \in W \text{ and } z \in Z \end{cases}$$

[4] Indeed, achieving multiplicative compatibility is *not* possible for odd n over local fields $\mathbb{R}$ or $\mathbb{Q}_p$, for non-trivial reasons.

[5] Thus, often, one finds the vector space I_W identified with L^2 on a fixed complementary Lagrangian subspace, although this presents some hazards.

We have

$$\big(T_{W\to W'}\delta_W\big)(h) \;=\; \frac{1}{\#(W'\cap W)}\sum_{u\in W'}\delta_W(e^u\cdot h) \;=\; \frac{1}{\#(W\cap W')}\sum_{u\in W':e^u h\in\exp(W+Z)}\omega(e^u h)$$

Taking $h=1\in H$,

$$\big(T_{W\to W'}\delta_W\big)(1) \;=\; \frac{1}{\#(W\cap W')}\sum_{u\in W'}\delta_W(e^u) \;=\; \frac{1}{\#(W\cap W')}\sum_{u\in W':e^u\in\exp(W+Z)}\omega(e^u) \;=\; 1$$

Thus, $T_{W\to W'}$ is non-zero.

Fix Lagrangian W, and as a naive approximation to a Segal-Shale-Weil/oscillator representation $\rho^{\rm nf}$ on I_W take $\rho^{\rm nf}(g)$ to be the composition

$$\rho^{\rm nf}(g) = T_{W^g\to W}\circ\tau_g \;:\; I_W\longrightarrow I_W^g = I_{W^g}\longrightarrow I_W$$

There is no guarantee that this normalizes-by-constants to achieve multiplicativity $\rho^{\rm nf}(xy)=\rho^{\rm nf}(x)\circ\rho^{\rm nf}(y)$. In fact, it does not, but, with this as starting point, we can hope to *adjust* to arrange $\rho(xy)=\rho(x)\circ\rho(y)$. Multiplicativity holds *up to scalars*. Therefore, to determine the (scalar!) discrepancy, if any, it suffices to examine the behavior of a single function, such as δ_W, evaluated at a single point, for example 1_H.

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5. *The simplest case:* $SL_2(\mathbb{F}_q)$

Let $Q\approx\mathbb{F}_q^n$ be an n-dimensional non-degenerate quadratic space. Take $V=Q\otimes\mathbb{F}_q^2$ where $\mathbb{F}_q^2$ has the usual alternating form

$$\langle (x,y),\,(x',y')\rangle \;=\; xy'-x'y \qquad\qquad (\text{for } x,y,x',y'\in\mathbb{F}_q)$$

This gives a natural imbedding $O(Q)\times SL_2(\mathbb{F}_q)\subset Sp(V)$. Let e_1,e_2 be the standard basis of $\mathbb{F}_q^2$, and $W_i=\mathbb{F}_q\cdot e_i$. Use Lagrangian subspace $W=Q\otimes W_2\subset V$ and $W'=Q\otimes W_1$, and coordinates $(u,v,z)\in\mathfrak{h}$ with $u\in W'$, $v\in W$, and $z\in\mathbb{F}_q$. In these coordinates, $G=SL_2(\mathbb{F}_q)$ acts by matrix multiplication on u,v and trivially on the center:

$$\tau_x(u,v,z) \;=\; (u,v,z)\cdot x \;=\; (u,v,z)\begin{pmatrix}a&b\\c&d\end{pmatrix} \;=\; (ua+vc,\,ub+vd,\,z) \qquad\quad \Big(\text{for } x=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{F}_q)\Big)$$

The standard (upper-triangular) parabolic $P\subset G$ stabilizes W_2, and, therefore, W. The Bruhat decomposition has just two cells: $SL_2(\mathbb{F}_q)=P\cup Pw_oP$. For $x\in SL_2(\mathbb{F}_q)$,

$$W^x\cap W \;=\; \begin{cases} W & \text{for } x\in P \\ \{0\} & \text{for } x\notin P\end{cases}$$

[5.1] Naive approximation to the Segal-Shale-Weil/oscillator representation Our naive approximation to a Segal-Shale-Weil/oscillator representation $\rho^{\rm nf}$ of G on $I_W={\rm Ind}_{W\oplus Z}^H\omega$ is

$$\begin{cases} \rho^{\rm nf}(x) \;=\; \tau_x & (\text{for } x\in P) \\ \rho^{\rm nf}(x) \;=\; T_{W^x\to W}\circ\tau_x & (\text{for } x\notin P)\end{cases}$$

[5.2] The big cell Even though $SL(2(\mathbb{F}_q)$ is *finite*, there is considerable sense in construing *generic* elements of $SL_2(\mathbb{F}_q)$ as lying in the big cell Pw_oP. The heuristic is that adjusting constants to achieve multiplicative compatibility for x,y,xy all in the big cell should suggest correct constant adjustment on the whole $SL_2(\mathbb{F}_q)$.

For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q)$ with $c \neq 0$,

$$\Big(\rho^{\mathrm{nf}}(x)f\Big)(u,0,0) \;=\; (T_{W^x \to W}\,\tau_x f)(u,0,0) \;=\; \sum_{v \in \mathbb{F}_q^n} (\tau_x f)\Big((0,v,0)\cdot(u,0,0)\Big)$$

$$= \sum_{v \in \mathbb{F}_q^n} (\tau_x f)\Big((0,0,-\frac{uv}{2})\cdot(u,v,0)\Big) \;=\; \sum_{v \in \mathbb{F}_q^n} \omega\Big(-\frac{uv}{2}\Big)\cdot f(ua+vc, ub+vd, 0)$$

With $f = \delta_W$ the only non-zero summand is where $ua + vc = 0$, giving $v = -ua/c$, and

$$(T_{W^x \to W}\tau_x \delta_W)(u,0,0) \;=\; \omega\Big(\frac{u^2 a}{2c}\Big)$$

The value at 1_H is

$$(T_{W^x \to W}\tau_x \delta_W)(0,0,0) \;=\; 1$$

For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $y = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $c, c', c'' \neq 0$, let

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} \;=\; xy \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

Compute the composition of $T_{W^x \to W} \circ \tau_x$ and $T_{W^y \to W} \circ \tau_y$ on δ_W evaluated at 1_H:

$$(T_{W^x \to W}\,\tau_x T_{W^y \to W}\,\tau_y \delta_W)(0,0,0) \;=\; \sum_{u \in \mathbb{F}_q^n} (\tau_x T_{W^y \to W}\,\tau_y \delta_W)(0,u,0)$$

$$= \sum_{u \in \mathbb{F}_q^n} (T_{W^y \to W}\,\tau_y \delta_W)\Big((0,u,0)x\Big) \;=\; \sum_{v \in \mathbb{F}_q^n}\sum_{u \in \mathbb{F}_q^n} (\tau_y \delta_W)\Big((0,v,0)\cdot(0,u,0)x\Big)$$

$$= \sum_{v \in \mathbb{F}_q^n}\sum_{u \in \mathbb{F}_q^n} \delta_W\Big((0,v,0)y\cdot(0,u,0)xy\Big) \;=\; \sum_{v \in \mathbb{F}_q^n}\sum_{u \in \mathbb{F}_q^n} \delta_W\Big((vc', vd', 0)\cdot(uc'',\, ud'',\, 0)\Big)$$

$$= \sum_{v \in \mathbb{F}_q^n}\sum_{u \in \mathbb{F}_q^n} \delta_W\Big(vc' + uc'',\, vd' + ud'',\, -\frac{vc'\cdot ud''}{2} + \frac{uc''\cdot vd'}{2}\Big)$$

The first argument of δ_W must be 0 to give a non-zero value, so the sum is over u, v such that $vc' + uc'' = 0$. Replacing v by $-uc''/c'$ gives

$$(T_{W^x \to W}\,\tau_x T_{W^y \to W}\,\tau_y \delta_W)(0,0,0) \;=\; \sum_{u \in \mathbb{F}_q^n} \delta_W\Big(0,\, \frac{-uc''}{c'}d' + ud'',\, \frac{uc''\cdot ud''}{2} - \frac{uc''\cdot uc''d'}{2c'}\Big)$$

$$= \sum_{u \in F_q^n} \omega\Big(u^2 \cdot \frac{c''(c'd'' - c''d')}{c'}\Big)$$

Expanding and simplifying,

$$c'd'' - c''d' \;=\; c'(cb' + dd') - (ca' + dc')d' \;=\; c(b'c' - a'd') \;=\; -c$$

Thus,

$$(T_{W^x \to W}\,\tau_x T_{W^y \to W}\,\tau_y \delta_W)(0,0,0) \;=\; \sum_{u \in F_q^n} \omega\Big(-u^2\cdot\frac{c''c}{c'}\Big)$$

Meanwhile, we saw that

$$(T_{W^{xy}\to W}\,\tau_{xy}\,\delta_W)(0,0,0) \;=\; 1$$

Thus, the (scalar) discrepancy between $\rho^{\rm nf}(xy)$ and $\rho^{\rm nf}(x)\circ\rho^{\rm nf}(y)$, for x,y,xy in the big cell, is

$$\rho^{\rm nf}(x)\circ\rho^{\rm nf}(y) \;=\; T_{W^x\to W}\,\tau_x T_{W^y\to W}\,\tau_y$$

$$= \sum_{u\in F_q^n} \omega\Big(-u^2\cdot\frac{c''c}{c'}\Big)\cdot T_{W^{xy}\to W}\,\tau_{xy} \;=\; \sum_{u\in F_q^n} \omega\Big(-u^2\cdot\frac{c''c}{c'}\Big)\cdot \rho^{\rm nf}(xy)$$

We want to adjust all the $\rho^{\rm nf}(x)$ by constants to recover multiplicativity.

Since the groundfield is $\mathbb{F}_q$, as opposed to $\mathbb{R}$ or $\mathbb{Q}_p$, the quadratic exponential sums are nearly *characters* on $\mathbb{F}_q^\times$. Specifically, with $\chi:\mathbb{F}_q^\times\to\{1,-1\}$ by assigning value $+1$ for non-zero squares and -1 for non-squares,

$$\sum_{u\in\mathbb{F}_q}\omega(-u^2\cdot\alpha) \;=\; \chi(\alpha)\cdot\sum_{u\in\mathbb{F}_q}\omega(-u^2) \qquad\qquad (\text{for } \alpha\in\mathbb{F}_q^\times)$$

In the n-dimensional situation, since quadratic forms are diagonalizable,

$$\sum_{u\in\mathbb{F}_q^n}\omega(-u^2\cdot\alpha) \;=\; \chi(\alpha)^n\cdot\sum_{u\in\mathbb{F}_q^n}\omega(-u^2) \qquad\qquad (\text{for } \alpha\in\mathbb{F}_q^\times)$$

Thus, for x,y,xy all in the big cell,

$$\rho^{\rm nf}(x)\circ\rho^{\rm nf}(y) \;=\; \chi(c'')^n\cdot\chi(c)^n\cdot\chi(c')^n\cdot\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)\cdot\rho^{\rm nf}(xy)$$

Rearranging, certainly

$$\chi(c)^n\rho^{\rm nf}(x)\circ\chi(c')^n\rho^{\rm nf}(y) \;=\; \sum_{u\in\mathbb{F}_q^n}\omega(-u^2)\cdot\chi(c'')^n\rho^{\rm nf}(xy)$$

or

$$\frac{\chi(c)^n\rho^{\rm nf}(x)}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)}\circ\frac{\chi(c')^n\rho^{\rm nf}(y)}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)} \;=\; \frac{\chi(c'')^n\rho^{\rm nf}(xy)}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)}$$

This is the same as

$$\frac{\rho^{\rm nf}(x)}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2\cdot c)}\circ\frac{\rho^{\rm nf}(y)}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2\cdot c')} \;=\; \frac{\rho^{\rm nf}(xy)}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2\cdot c'')}$$

Thus, we have the desired multiplicative compatibility $\rho(x)\rho(y)=\rho(xy)$ for x,y,xy in the big cell, by taking

$$\rho(x) \;=\; \frac{\rho^{\rm nf}(x)}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2\cdot c)} \;=\; \frac{T_{W^x\to W}\circ\tau_x}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2\cdot c)} \;=\; \frac{\chi(c)^n\,T_{W^x\to W}\circ\tau_x}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)} \qquad\qquad \Big(x=\begin{pmatrix}a&b\\c&d\end{pmatrix} \text{ and } c\neq 0\Big)$$

[5.3] The small cell On one hand, our naive normalization $\rho^{\rm nf}(x)$ for x in the small cell P is just τ_x. On the other hand, $x \in P$ can be written as a product of two elements from the big cell Pw_oP, and in several ways, for example, $x = -w_o \cdot w_o x$. Hopefully, any two such re-expressions *agree*.

The standard (diagonal) Levi component M of P acts on H by

$$\tau_x(u,v,z) \;=\; (u,v,z)\cdot x \;=\; (u,v,z)\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \;=\; (ua, va^{-1}, z) \qquad (\text{with } x = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \in M)$$

the unipotent radical N by

$$\tau_x(u,v,z) \;=\; (u,v,z)\cdot x \;=\; (u,v,z)\begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix} \;=\; (u, v+ut, z) \qquad (\text{with } x = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix} \in N)$$

and the long Weyl element by

$$\tau_{w_o}(u,v,z) \;=\; (u,v,z)\cdot w_o \;=\; (u,v,z)\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \;=\; (v, -u, z) \qquad (\text{with } w_o = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix})$$

Using the commutation rules, in exponential coordinates the action of the unipotent radical is

$$\tau_x(u,0,0) \;=\; (u, ut, 0) \;=\; (0, ut, \frac{t}{2}u^2)\cdot(u,0,0) \qquad (\text{with } x = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix})$$

where u^2 is the value of the quadratic form on u. Thus, on $f \in I_W$, up to constants the Levi component acts by dilation and the unipotent radical by multiplication by a quadratic exponential:

$$\begin{cases} \tau_x f(u,0,0) \;=\; f(ua,0,0) & (\text{for } x = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}) \\ \tau_x f(u,0,0) \;=\; \omega(\frac{t}{2}u^2)\cdot f(u,0,0) & (\text{for } x = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}) \end{cases}$$

Although $x \to \tau_x$ is already a representation of P on I_W, adjustment of constants is required for compatibility with the action of the big cell.

The big-cell computations suggest the correct renormalization-by-constants for elements in the small Bruhat cell P. For example, $x = -w_o \cdot w_o x$ expresses $x \in P$ as a product of elements from the big cell, so we might *hope* to prescribe a genuine representation ρ by taking

$$\rho(x) \;=\; \rho(-w_o)\circ\rho(w_o x) \;=\; \frac{\chi(-1)^n\, T_{W^{-w_o}\to W}\circ\tau_{-w_o}}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)} \circ \frac{\chi(a)^n\, T_{W^{w_o x}\to W}\circ\tau_{w_o x}}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)} \qquad (\text{for } x\in P)$$

By uniqueness, this differs by a constant from $\rho^{\rm nf}(x) = \tau_x$, and the constant can be ascertained by evaluation on δ_W at 1_H: for $x = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$,

$$\Big(T_{W^{-w_o}\to W}\circ\tau_{-w_o}\circ T_{W^{w_o x}\to W}\circ\tau_{w_o x}\,\delta_W\Big)(0,0,0) \;=\; \sum_{u\in\mathbb{F}_q^n}\Big(\tau_{-w_o}\circ T_{W^{w_o x}\to W}\circ\tau_{w_o x}\,\delta_W\Big)(0,u,0)$$

$$=\; \sum_{u\in\mathbb{F}_q^n}\Big(T_{W^{w_o x}\to W}\circ\tau_{w_o x}\,\delta_W\Big)(-u,0,0) \;=\; \sum_{v\in\mathbb{F}_q^n}\sum_{u\in\mathbb{F}_q^n}\Big(\tau_{w_o x}\,\delta_W\Big)\Big((0,v,0)\cdot(-u,0,0)\Big)$$

$$=\; \sum_{v\in\mathbb{F}_q^n}\sum_{u\in\mathbb{F}_q^n}\Big(\tau_{w_o x}\,\delta_W\Big)\Big(-u, v, \frac{uv}{2}\Big) \;=\; \sum_{v\in\mathbb{F}_q^n}\sum_{u\in\mathbb{F}_q^n}\delta_W(va, ua^{-1}, \frac{uv}{2})$$

$$=\; \sum_{u\in\mathbb{F}_q^n}\delta_W(0, ua^{-1}, 0) \;=\; q^n \;=\; q^n\cdot(\tau_x\delta_W)(0,0,0)$$

Replacing the normalizing constants, we are hoping that it is sufficient to take

$$\rho(x) \;=\; \frac{\chi(-1)^n}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)}\cdot\frac{\chi(a)^n}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)}\cdot q^n\cdot\tau_x \qquad\qquad (\text{for } x\in P)$$

An immediate sensibility check is taking $x = 1_2$, that is, $a = 1$, which should produce $\rho(1) = 1$: indeed, reducing to the one-dimensional case by diagonalizing the quadratic form, it is standard that

$$\Big(\sum_{u\in\mathbb{F}_q}\omega(-u^2)\Big)^2 \;=\; \sum_{u,v\in\mathbb{F}_q}\omega(-u^2-v^2) \;=\; \chi(-1)\cdot q$$

This identity simplifies the presentation of $\rho(x)$ on the Levi component, to

$$\rho(x) \;=\; \chi(a)^n\cdot\tau_x \qquad\qquad (\text{for } x = \begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix})$$

For $x = \begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix}$ in the unipotent radical, a similar computation gives

$$\Big(T_{W^{-w_o}\to W}\circ\tau_{-w_o}\circ T_{W^{w_o x}\to W}\circ\tau_{w_o x}\,\delta_W\Big)(0,0,0) \;=\; q^n\cdot(\tau_x\delta_W)(0,0,0)$$

Thus, even more simply than for the Levi component, we are taking

$$\rho(x) \;=\; \tau_x \qquad\qquad (\text{for } x = \begin{pmatrix} 1 & t\\ 0 & 1\end{pmatrix})$$

We see that this ρ restricted to P is a genuine representation on I_W, just slightly renormalized from τ_x itself:

$$\rho(x) \;=\; \chi(a)^n\cdot\tau_x \qquad\qquad (\text{for } x = \begin{pmatrix} a & *\\ 0 & a^{-1}\end{pmatrix})$$

Thus, the above heuristic fully suggests the re-normalization of constants to obtain a (hopefully genuine) representation ρ, given by

$$\rho(x) \;=\; \begin{cases} \chi(a)^n\cdot\tau_x & (\text{for } x = \begin{pmatrix} a & *\\ 0 & a^{-1}\end{pmatrix}\in P)\\[2ex] \dfrac{\chi(c)^n}{\sum_{u\in\mathbb{F}_q^n}\omega(-u^2)}\cdot T_{W^x\to W}\,\tau_x & (\text{for } x = \begin{pmatrix} * & *\\ c & *\end{pmatrix}\notin P)\end{cases}$$

Thus, with constants adjusted for multiplicative compatibility of ρ on the big cell, ρ is completely determined on the small cell.

[5.4] Remark: Thus, the heuristic suggests a complete specification of re-normalizing constants. Nevertheless, this is not quite a complete proof that ρ is a genuine representation. As presented, some other cases must be checked, which we do in the following section. Nevertheless, the point of the present section is a natural sequence of events completely *and correctly* specifying ρ.

[5.5] Toward complete certification

For a complete proof that ρ is a genuine representation, we must certify the multiplicative compatibililty $\rho(x)\circ\rho(y) = \rho(xy)$ for all cases of x, y, xy in the big or small cells.

With or without the multiplicative compatibility, we do have *associativity*: $\rho(x)(\rho(y)\rho(z)) = (\rho(x)\rho(y))\rho(z)$, which can be leveraged into a general argument for multiplicative compatibility, *after* we know that

$$\rho(x)\circ\rho(x^{-1}) \;=\; \rho(1) \;=\; 1_{I_W} \qquad\qquad (x \text{ in the big cell})$$

To verify the latter, we may as well consider the more general situation of x, y in the big cell, and $xy \in P$. We start from the point in the computation of $\rho(x)^{\rm nf} \circ \rho^{\rm nf}(y)$ on δ_W at 1_H, where xy had not yet been assumed in the big cell. Namely, letting c, c', c'' be the lower left entries of x, y, xy, we had

$$\Big(\rho(x)^{\rm nf} \circ \rho^{\rm nf}(y)\, \delta_W\Big)(0,0,0) \;=\; \sum_{u \in F_q^n} \omega\Big(-u^2 \cdot \frac{c'' c}{c'}\Big)$$

Since $c'' = 0$ for $xy \in P$,

$$\Big(\rho(x)^{\rm nf} \circ \rho^{\rm nf}(y)\, \delta_W\Big)(0,0,0) \;=\; q^n \qquad\qquad (\text{for } xy \in P)$$

Meanwhile,

$$\Big(\rho^{\rm nf}(xy)\delta_W\Big)(0,0,0) \;=\; \Big(\tau_x y \delta_W\Big)(0,0,0) \;=\; \delta_W(0,0,0) \;=\; 1 \qquad\qquad (\text{for } xy \in P)$$

Thus,

$$\rho(x)^{\rm nf} \circ \rho^{\rm nf}(y) \;=\; q^n \cdot \rho^{\rm nf}(xy)$$

Thus, for $xy \in P$,

$$\rho(x)\rho(y) \;=\; \frac{\rho^{\rm nf}(x)}{\sum_{u\in\mathbb{F}_q^n} \omega(-u^2\cdot c)} \circ \frac{\rho^{\rm nf}(y)}{\sum_{u\in\mathbb{F}_q^n} \omega(-u^2\cdot c')} \;=\; \frac{q^n \cdot \rho^{\rm nf}(xy)}{\sum_{u\in\mathbb{F}_q^n} \omega(-u^2\cdot c)\cdot \sum_{u\in\mathbb{F}_q^n} \omega(-u^2\cdot c')}$$

Since $xy \in P$, the $\rho(xy)$ renormalized as earlier differs from $\rho^{\rm nf}(xy)$ only by $\chi(a'')^n$, where

$$\begin{pmatrix} a'' & b'' \\ 0 & (a'')^{-1} \end{pmatrix} \;=\; xy \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \;=\; \begin{pmatrix} aa' + bc' & * \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

Thus, $ca' + dc' = 0$, which is $a' = -dc'/c$, so

$$a'' \;=\; aa' + bc' \;=\; a \cdot \frac{-dc'}{c} + bc' \;=\; \frac{(-ad + bc)\cdot c'}{c} \;=\; -\frac{c'}{c}$$

Thus,

$$\chi(a'') \;=\; \frac{\chi(-1)\cdot\chi(c)}{\chi(c')} \;=\; \chi(-1)\cdot\chi(c)\cdot\chi(c')$$

Thus, as hoped,

$$\rho(x)\rho(y) \;=\; \frac{q^n \cdot \rho^{\rm nf}(xy)}{\sum_{u\in\mathbb{F}_q^n} \omega(-u^2\cdot c)\cdot \sum_{u\in\mathbb{F}_q^n} \omega(-u^2\cdot c')} \;=\; \frac{\chi(c)^n \chi(c')^n \cdot q^n \cdot \rho^{\rm nf}(xy)}{\sum_{u\in\mathbb{F}_q^n} \omega(-u^2)\cdot \sum_{u\in\mathbb{F}_q^n} \omega(-u^2)}$$

$$=\; \frac{\chi(c)^n \chi(c')^n \cdot q^n \cdot \rho^{\rm nf}(xy)}{\chi(-1)^n \cdot q^n} \;=\; \chi(a'')^n \cdot \rho^{\rm nf}(xy) \;=\; \rho(xy) \qquad\qquad (\text{for } x, y \notin P \text{ and } xy \in P)$$

In particular, as hoped,

$$\rho(x) \circ \rho(x^{-1} \;=\; \rho(1) \qquad\qquad (\text{for } x \text{ in the big cell})$$

[5.6] Remaining certifications We have verified that $\rho(x)\rho(y) = \rho(xy)$ when x, y are in the big cell, whether xy lies in the big cell or in the small cell. The multiplicative compatibility is also visible for both $x, y \in P$ when they are expressed as products from the big cell, and the outcome is examined in detail.

It remains to treat the case where exactly one of x or y is in the big cell, one in the small cell. This *could* be done by repeating the same sort of calculations as above. However, especially as prototype for analogous computations for larger groups, there is an easier argument, that plays on the idea that generic elements of the group are in the big cell, and uses the big-cell computations already done.

Namely, given $x, y \in SL_2(\mathbb{F}_q)$, let ε be in the big cell, such that $x\varepsilon$ and $\varepsilon^{-1}y$ are in the big cell. Because $\rho(x)$ is unambiguously defined as $\rho(x) = \rho(a)\rho(b)$ for any a, b in the big cell so that $x = ab$, we have

$$\rho(x) \circ \rho(y) \;=\; \Big(\rho(x\varepsilon)\rho(\varepsilon^{-1}\Big)\Big(\rho(\varepsilon)\rho(\varepsilon^{-1}y)\Big) \;=\; \rho(x\varepsilon)(\varepsilon^{-1}y) \;=\; \rho\Big((x\varepsilon)(\varepsilon^{-1}y)\Big) \;=\; \rho(xy)$$

This completes the verification that ρ is a genuine representation of $SL_2(\mathbb{F}_q)$.

[5.7] Further comments

Again, the natural appearance of a Segal-Shale-Weil/oscillator representation is as a *projective* representation, with natural ambiguity of the constants.

When it is *possible* to choose constants to achieve multiplicative compatibility on $SL_2(k)$, as opposed to a metaplectic two-fold cover, it is possible to present a correct description of the representation as a *fact-to-be-verified*, without indication of how it could be *discovered*. This is methodologically unsatisfying, but does dodge the discussion of Heisenberg groups and Stone-vonNeumann theorems. However, without the discussion of Heisenberg groups, it is difficult to see that the only obstacle is compatible choice of *constants*. That is, in that seemingly simpler context, the issue of multiplicative compatibility must be addressed for *arbitrary* vectors in the representation space. This is a *feasible* computation, but misleadingly portrays the degree to which the multiplicativity might fail, and is more difficult than the simplified and reasonably ordered computation above.

For groups $Sp_n(\mathbb{F}_q)$ with larger n, the technical device of specifying a compatible family of adjustments-by-constants by writing an arbitrary group element as a product of big-cell elements succeeds, as in the simplest case above. A slightly generalized Bruhat decomposition not with respect to a *minimal* parabolic, but with respect to a parabolic fixing a maximal Lagrangian subspace W (sometimes called a *Siegel parabolic*), has cells characterized by $\dim(W \cap gW)$. The general form of the argument is the same.

Over local fields k such as $\mathbb{R}$ and $\mathbb{Q}_p$, the behavior of the quadratic exponential sums is only slightly less congenial. *Squares* are no longer of index 2 in $k^\times$, unlike the finite-field case. Instead, *norms* from quadratic field extensions are of index 2, and the corresponding quadratic exponential sums are essentially the *norm residue characters*. Thus, the projective form of the representation for $SL_2(\mathbb{R})$ and $SL_2(\mathbb{Q}_p)$ can be adjusted to give a genuine representation only for $V = Q \otimes k^2$ with $\dim_k \mathbb{Q}$ *even*.

Further, over local fields, the proof of the Stone-vonNeumann is correspondingly less trivial, since genuine analytical issues come into play.

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6. *Theta correspondence for* $O(1) \times SL_2(\mathbb{F}_q)$

The smallest alternating space $Q \otimes \mathbb{F}_q^2$ has $Q = \mathbb{F}_q$ and the quadratic form is literally squaring: $u \cdot u = u^2$. The representation space is $I_W = \mathrm{Ind}_{W \oplus Z}^H \omega$ with $W = Q \otimes \mathbb{F}_q e_2 \approx \mathbb{F}_q e_2$, which as a complex vector space is identifiable with $L^2(W')$ with $W' = Q \otimes \mathbb{F}_q e_1 \approx \mathbb{F}_q e_1$. The space I_W is q-dimensional.

[6.1] Theta lifts from $O(1)$ to $SL_2(\mathbb{F}_q)$ The group $O(1)$ has two representations, the trivial one, and the unique non-trivial ± 1-valued representation. The action of $O(1)$ on functions on $W' = Q \otimes \mathbb{F}_q e_1$ is linear, by $(\pm 1) f(u, 0, 0) = f(\pm u, 0, 0)$.

The *theta lift* of a representation π of $O(1)$ is the π-isotype in I_W, as a representation of $SL_2(\mathbb{F}_q)$.

The theta lift of the trivial representation of $O(1)$ to $SL_2(\mathbb{F}_q)$ is the restriction ρ_+ of ρ to *even* functions on $W' \approx \mathbb{F}_q$, and the *theta-lift* of the ± 1 representation of $O(1)$ is the restriction ρ_- of ρ to *odd* functions on $W' \approx \mathbb{F}_q$. The restrictions ρ_+ and ρ_- are $\frac{q+1}{2}$-dimensional and $\frac{q-1}{2}$-dimensional, respectively.

Dimension is sufficient to identify these representations in the classification of irreducibles of $SL_2(\mathbb{F}_q)$: there are exactly two irreducibles of dimension $\frac{q+1}{2}$ of $SL_2(\mathbb{F}_q)$, namely, the irreducible summands of the *irregular principal series* $I_{\pm 1}$ induced from the non-trivial ± 1-valued character on P. [6] Thus, ρ_+ is one of these. There is a unique *supercuspidal* representation of $GL_2(\mathbb{F}_q)$ anomalously decomposing as a direct sum of two $\frac{q-1}{2}$-dimensional irreducibles when restricted to $SL_2(\mathbb{F}_q)$. [7] Thus, ρ_- is one of these two summands.

While happy to know the outcomes, we might want identifications depending less on the finite-dimensionality.

[6.2] Jacquet modules The Jacquet modules $J_N\rho_\pi$ are identifiable with N-fixed functions in I_W. The action of N is by multiplication by $\omega(\frac{1}{2}u^2t)$ for varying $t \in \mathbb{F}_q$ and $u \in W'$. Thus, the Jacquet module is identifiable with functions in I_W viewed as functions on W' supported just on $\{0\}$. This is a one-dimensional space, and $x = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in M$ acts on such functions by

$$(\rho(x)f)(0,0,0) \;=\; (\tau_x f)(0,0,0) \;=\; \chi(a)\cdot f(0,0,0)$$

Thus, as M-module, the Jacquet module of ρ_+ is $J_N\rho_+ \approx \chi$, while $J_N\rho_- \approx \{0\}$. Thus, the non-supercuspidal (and non-trivial) part of ρ_+ is indeed one of the summands of the (irregular) χ^{th} principal series, while ρ_- has no non-supercuspidal part.

[6.3] Whittaker/Gelfand-Graev models From a non-trivial additive character ψ form the corresponding Whittaker/Gelfand-Graev space

$$W_\psi \;=\; \mathrm{Ind}_N^{SL_2(\mathbb{F}_q)}\psi$$

A natural intertwining S_ψ from the whole $\rho = \rho_+ \oplus \rho_-$ to the Whittaker space is

$$(S_\psi f)(x) \;=\; \frac{1}{q}\sum_{t\in\mathbb{F}_q} \psi(-t)\,\rho(nx)f \qquad\qquad \text{(where } n = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ and } x \in SL_2(\mathbb{F}_q))$$

An image $S_\psi f(x)$ is a function on H. At $x = 1 \in SL_2(\mathbb{F}_q)$, this function is still in I_W, and

$$S_\psi f(1)(u,0,0) \;=\; \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-t)\,(\tau_{n_t}f)(u,0,0) \;=\; \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-t)\,f\big((u,0,0)n_t\big)$$

$$=\; \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-t)\,f(u,ut,0) \;=\; \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-t)\,f\Big((0,ut,\tfrac{1}{2}u^2t)\cdot(u,0,0)\Big)$$

$$=\; \frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-t)\,\omega(\tfrac{1}{2}u^2t)\cdot f(u,0,0) \;=\; f(u,0,0)\cdot\frac{1}{q}\sum_{t\in\mathbb{F}_q}\psi(-t)\,\omega(\tfrac{1}{2}u^2t)$$

This is 0 unless $\psi(t) = \omega(\frac{1}{2}u^2t)$, in which case it is $f(u,0,0)$. For example, taking $\psi(t) = \omega(\frac{1}{2}v^2t)$ for $v \in \mathbb{F}_q^\times$ guarantees that the intertwining S_ψ is not the 0-map.

We recall that both $\frac{q+1}{2}$-dimensional summands of the irregular principal series for $SL_2(\mathbb{F}_q)$ have a Whittaker model for one of the two inequivalent characters, but *not* for the other.

Similarly, both $\frac{q-1}{2}$-dimensional summands of the supercuspidal representation of $GL_2(\mathbb{F}_q)$ that properly decomposes upon restriction to $SL_2(\mathbb{F}_q)$ have a Whittaker model for one character, but *not* for the other.

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[6] This irregular principal series is the restriction to $SL_2(\mathbb{F}_q)$ of a *regular* (hence, irreducible) principal series of $GL_2(\mathbb{F}_q)$. Principal series for $GL_2(\mathbb{F}_q)$ whose two characters whose ratio is not $\{\pm 1\}$-valued remain irreducible upon restriction to $SL_2(\mathbb{F}_q)$.

[7] All the other supercuspidals of $GL_2(\mathbb{F}_q)$ remain irreducible when restricted to $SL_2(\mathbb{Q}_p)$.

Thus, neither of $\rho_\pm$ has a Whittaker model for additive character *inequivalent* to $\omega(\frac{1}{2}t)$. Without *some* further argument, such as dimension-count, this does not quite prove that $\rho_\pm$ is *exactly* as indicated, but only that no *other* irreducibles can appear.

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7. *Theta correspondence for isotropic* $O(2) \times SL_2(\mathbb{F}_q)$

With q odd, on $\mathbb{F}_q^2$ there are exactly two non-isomorphic non-degenerate quadratic space structures Q, namely, the (isotropic) $(x,y) \to xy$ and, up to constant multiples, the (anisotropic) *norm* from the unique quadratic field extension of $\mathbb{F}_q$.

In both cases, $O(Q)$ is a dihedral group, since its index over the special orthogonal group $SO(Q)$ is 2, and

$$SO(Q) \;\approx\; \begin{cases} \mathbb{F}_q^\times & \text{(for } Q \text{ isotropic)} \\ \{\alpha \in \mathbb{F}_{q^2}^\times : N(\alpha) = 1\} & \text{(for } Q \text{ anisotropic)} \end{cases} \qquad \text{(where } N = N_{\mathbb{F}_{q^2}/F_q}\text{)}$$

Thus, the two-dimensional irreducibles π of $O(Q)$, when restricted to $SO(Q)$, are of the form $\pi \approx \pi_1 \oplus \pi_1^\vee$ for a one-dimensional representation π_1 of $SO(Q)$ with $\pi_1^\vee(\theta) = \pi_1(\theta^{-1}) \neq \pi_1(\theta)$ for $\theta \in SO(Q)$. In either case, given an irreducible π of $O(Q)$, the π-isotype ρ_π inside ρ, as a representation of $SL_2(\mathbb{F}_q)$, is the *theta-lift* of π.

Here we consider the isotropic case.

[7.1] Jacquet modules The Jacquet module $J_N\rho_\pi$ of ρ_π can be computed as the $\rho_\pi(N)$-fixed-points on I_W. Since $\rho(N)$ acts by multiplication by $\omega(\frac{1}{2}u^2t)$, where $u^2 = (x,y)^2 = xy$ is the quadratic form on $W' \approx \mathbb{F}_q^2$, the Jacquet module is the collection of functions f in I_W with support on the 0-set of Q, namely, the union U of the two axes.

There are exactly two $O(Q)$-orbits on U, namely, the origin $(0,0)$, and the complement $U' = U - \{(0,0)\}$ of the origin. For non-trivial π, the support of $SO(Q), \pi$-equivariant $f \in I_W$ cannot include the origin. Since $SO(Q)$ is *simply* transitive on each axis, such f is determined up to a constant on each axis. Indeed, since $SO(Q)$ includes maps $a_t : (x,y) \to (tx, t^{-1}y)$,

$$f\big((t,0),0,0\big) \;=\; \pi_1(a_t)\cdot f\big((1,0),0,0\big) \qquad\qquad f\big((0,t^{-1}),0,0\big) \;=\; \pi_1(a_t)\cdot f\big((0,1),0,0\big)$$

In particular, as M-representation, with f supported on U', with $m = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$,

$$(\rho(m)f)((t,0),0,0) \;=\; (\tau_m f)((t,0),0,0) \;=\; f((ta,0),0,0) \;=\; \pi_1(a)\cdot f((t,0),0,0)$$

Similarly,

$$(\rho(m)f)((0,t),0,0) \;=\; (\tau_m f)((0,t),0,0) \;=\; f((0,ta),0,0) \;=\; \pi_1(a^{-1})\cdot f((0,t),0,0)$$

Thus, the Jacquet module of ρ_π is two-dimensional:

$$J_N\,\rho_\pi \;\approx\; \pi_1 \oplus \pi_1^\vee$$

For $\pi_1 \not\approx \pi_1^\vee$, this Jacquet module is uniquely identifiable as the Jacquet module of the π_1^{th} *principal series* representation of $SL_2(\mathbb{F}_q)$. This *suggests* that ρ_π is that principal series, although all that is *proven* is that there are no other non-supercuspidal summands.

By direct observation, $SO(Q)$ is *transitive* on level sets $\{u \in W' : u^2 = c\}$ for $c \neq 0$. Thus, the values of f in the π-isotype is uniquely determined up to a constant on each such level set with $c \neq 0$, while the values on the two axes are independent of each other. Thus, $\dim \rho_\pi = (q-1)+2 = q+1$. This is the dimension of a principal series, so ρ_π is precisely the π_1^{th} principal series of $SL_2(\mathbb{F}_q)$.

[7.2] Remark: Although it is more difficult to precisely exclude any possibility of supercuspidal summands, the fashion in which Jacquet modules can be determined is a prototype for the analogous problem over local fields.

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8. *Theta correspondence for anisotropic* $O(Q) \times SL_2(\mathbb{F}_q)$

Now consider *anisotropic* two-dimensional quadratic space Q. Up to a constant multiple, the quadratic form is the *norm* from the unique quadratic extension of $\mathbb{F}_q$.

[8.1] Jacquet modules The Jacquet module $J_N\rho_\pi$ of the theta lift ρ_π of π consists of functions in I_W supported on the isotropic vectors, now just $\{0\} \subset W'$. For non-trivial character π_1 of the norm-one subgroup of $\mathbb{F}_{q^2}^\times \approx SO(Q)$, no function supported on $\{0\}$ can be $SO(Q), \pi_1$-equivariant. Thus, the Jacquet module of ρ_π is $\{0\}$, so ρ_π is *supercuspidal.*

For trivial π of $O(Q)$, the Jacquet module of ρ_π is the one-dimensional space of functions supported on isotropic vectors, that is, supported at $0 \in W' \approx \mathbb{F}_q^2$. The Levi component M acts trivially. The trivial representation and the special subrepresentation of the principal series induced from the trivial representation of M both have one-dimensional Jacquet modules with M acting trivially.

[8.2] Dimension of ρ_π

We observe that $SO(Q)$ is transitive on $\{u \in W' : u^2 = c\}$ for each fixed $c \in \mathbb{F}_q$. Using the finite-dimensionality, with $\pi_1 \not\approx \pi_1^\vee$, this implies that $\dim \rho_\pi$ is at most $(q-1)$-dimensional, since f in the π-isotype must vanish for $u^2 = 0$. Thus, ρ_π is almost surely a supercuspidal irreducible.

The only possible complication would be that ρ_π were the direct sum of the two anomalous $\frac{q-1}{2}$-dimensional supercuspidal irreducibles. We neglect this point.

To prove that the supercuspidal representations of $SL_2(\mathbb{F}_q)$ thereby produced are *distinct*, we determine and compare *kernels* $K(u,v)$ for $\rho_\pi(x)$, meaning that, with fixed $x \in SL_2(\mathbb{F}_q)$,

$$(\rho_\pi(x)f)(u,0,0) \;=\; \sum_{v \in \mathbb{F}_q^2} K(u,v)\, f(v,0,0)$$

For example,

$$\mathrm{tr}\rho_\pi(x) \;=\; \sum_{v \in \mathbb{F}_q^2} K(v,v)$$

For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0$, and letting $\gamma = \sum_{t \in \mathbb{F}_q^2} \omega(-\frac{1}{2}t^2)$,

$$(\rho(x)f)(u,0,0) \;=\; (T_{W^x\to W}\tau_x\, f)(u,0,0) \;=\; \frac{1}{\gamma}\sum_{v\in\mathbb{F}_q^2}(\tau_x\, f)\Big((0,v,0)\cdot(u,0,0)\Big)$$

$$= \frac{1}{\gamma}\sum_{v\in\mathbb{F}_q^2}(\tau_x\, f)(u,v,\frac{-uv}{2}) \;=\; \frac{1}{\gamma}\sum_{v\in\mathbb{F}_q^2} f(ua+vc, ub+vd, \frac{-uv}{2})$$

$$= \frac{1}{\gamma}\sum_{v\in\mathbb{F}_q^2} f\Big((0, ub+vd, \frac{(ua+vc)(ub+vd)-uv}{2})\cdot(ua+vc,0,0)\Big)$$

$$= \frac{1}{\gamma}\sum_{v\in\mathbb{F}_q^2}\omega\Big(\frac{(ua+vc)(ub+vd)-uv}{2}\Big)\cdot f(ua+vc,0,0)$$

Change variables in the sum, replacing v by $(v-ua)/c$, to obtain

$$\frac{1}{\gamma}\sum_{v\in\mathbb{F}_q^2}\omega\Big(\frac{v(ubc+(v-ua)d)-u(v-ua)}{2c}\Big)\cdot f(v,0,0)$$

Using $ad-bc=1$, this is

$$\frac{1}{\gamma}\cdot\omega\Big(\frac{v^2d-2uv+u^2a}{2c}\Big)$$

This is the kernel for the whole $\rho(x)$. Composing with the projection to the π-isotype, the kernel for ρ_π is

$$\frac{1}{q+1}\sum_{\alpha\in SO(Q)}\Big(\pi_1(\alpha)+\pi_1^\vee(\alpha)\Big)\frac{1}{\gamma}\cdot\omega\Big(\frac{(\alpha v)^2d-2u\cdot\alpha v+u^2a}{2c}\Big)$$

Since α preserves the quadratic form $v\to v^2$, this is

$$K(u,v) \;=\; \frac{1}{\gamma(q+1)}\sum_{\alpha\in SO(Q)}\Big(\pi_1(\alpha)+\pi_1^\vee(\alpha)\Big)\,\omega\Big(\frac{v^2d-2u\cdot\alpha v+u^2a}{2c}\Big)$$

$$= \frac{1}{\gamma}\,\omega\Big(\frac{v^2d+u^2a}{2c}\Big)\cdot\frac{1}{q+1}\sum_{\alpha\in SO(Q)}\Big(\pi_1(\alpha)+\pi_1^\vee(\alpha)\Big)\,\omega\Big(\frac{-u\cdot\alpha v}{c}\Big)$$

The trace is

$$\mathrm{tr}\rho_\pi(x) \;=\; \sum_{v\in\mathbb{F}_q^2}K(v,v) \;=\; \frac{1}{\gamma}\sum_{v\in\mathbb{F}_a^2}\omega\Big(\frac{v^2(d+a)}{2c}\Big)\cdot\frac{1}{q+1}\sum_{\alpha\in SO(Q)}\Big(\pi_1(\alpha)+\pi_1^\vee(\alpha)\Big)\,\omega\Big(\frac{-v\cdot\alpha v}{c}\Big)$$

The bilinear form attached to the norm form from the quadratic extension is

$$u\cdot v \;=\; \tfrac{1}{2}(u^\sigma v+v^\sigma u) \qquad \text{(non-trivial Galois automorphism } \sigma)$$

so

$$\omega\Big(\frac{-v\cdot\alpha v}{c}\Big) \;=\; \omega\Big(\frac{-v^2(\alpha+\alpha^\sigma)}{2c}\Big)$$

We can exploit

$$\sum_{v\in\mathbb{F}_q^2}\omega(t\cdot v^2) \;=\; \begin{cases} q^2 & (\text{for } t=0) \\ -q & (\text{for } t\neq 0)\end{cases} \qquad (\text{for } t\in\mathbb{F}_q)$$

by judicious choices of x, for example,

$$x = \begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix}$$

With such x,

$$\mathrm{tr}\rho_\pi(x) = \frac{1}{\gamma(q+1)} \sum_{\alpha \in SO(Q)} \Big(\pi_1(\alpha) + \pi_1^\vee(\alpha)\Big) \sum_{v \in \mathbb{F}_q^2} \omega\Big(\frac{v^2(d - (\alpha + \alpha^\sigma))}{c}\Big)$$

Given $\beta \in SO(Q)$, $\beta \neq \pm 1$, take $d = \beta + \beta^\sigma$,

$$\mathrm{tr}\rho_\pi(x) = \frac{1}{\gamma(q+1)} \sum_{\alpha \in SO(Q)} \Big(\pi_1(\alpha) + \pi_1^\vee(\alpha)\Big) \begin{cases} q^2 & (\text{for } \alpha = \beta, \beta^\sigma) \\ -q & (\text{for } \alpha \neq \beta, \beta^\sigma) \end{cases} = \frac{2q}{\gamma}\Big(\pi_1(\beta) + \pi_1^\vee(\beta)\Big)$$

A similar outcome holds for $\beta = \pm 1$, without the 2 in the numerator.

Thus, for another representation $\pi' \approx \pi_2 \oplus \pi_2^\vee$ of $O(Q)$, with $\pi_2 \not\approx \pi_1, \pi_1^\vee$, for $\beta \in SO(Q)$ such that $\pi_1(\beta) \neq \pi_2(\beta), \pi_2^\vee(\beta)$, the traces $\mathrm{tr}\rho_\pi$ and $\mathrm{tr}\rho_{\pi'}$ differ at the corresponding x. Thus, the supercuspidal theta lifts ρ_π and $\rho_{\pi'}$ are mutually non-isomorphic for two-dimensional irreducibles π and π' of $O(Q)$.

[Gérardin 1977] P. Gérardin, *Weil representations associated to finite fields*, J. Algebra **46** (1977), 54-101.
