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08. *Local* $SL_2 \times O(2)$

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- Local Weil/oscillator representation
- Compatibility condition
- Explicit Bruhat decomposition
- Fourier transforms and quadratic exponentials

Weil's contribution to understanding *theta series* and related automorphic forms was explication of the representation theory in the situation. In particular, the representation theory shows that many features of a question about automorphic forms are, *local*, rather than *global*. Local questions are usually easier than global ones, so this improved understanding can render tractable seemingly intractable problems.

The presentation here is not in the style of Weil's original *Acta* paper, since we are giving the simplest possible example, and, in particular, avoid the important but technically complicating two-fold covers (*metaplectic groups*) necessary in general.

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1. *Local Weil/oscillator representation*

Let k be a local field (either $\mathbb{R}$ or $\mathbb{Q}_p$ or a finite extension of one of these) , K a quadratic extension of k (possibly $K \approx k \times k$), and ψ the standard additive character on k. Let N be the norm from K to k. Define a k-linear pairing on $K \times K$ by

$$\langle v, w\rangle = \frac{1}{4}[N(v+w) - N(v-w)]$$

so $\langle v, v\rangle = Nv$. Give k a Haar measure such that Fourier Inversion on K holds with ψ, this measure, and this pairing. When useful, we use $\mathscr{F}$ to denote Fourier transform. For $x \in k^\times$ and $v \in K$, let

$$\mu_x\varphi(v) = \varphi(vx)$$

denote the usual dilation operators.

For $x \in k$ and $v \in K$, let

$$S_x(v) = \psi(\frac{x}{2}\langle v, v\rangle)$$

Define the (local) Weil/oscillator representation of $SL(2,k)$ on Schwartz functions φ on K by

$$(n_x \cdot \varphi)(v) = S_x(v) \cdot \varphi(v)$$

$$(m_x \cdot \varphi)(v) = \nu(x)|x|^{1/2} \cdot (\mu_x)\varphi(v) = \nu(x)|x|^{1/2} \cdot \varphi(vx)$$

$$(w \cdot \varphi)(v) = c \cdot \widehat{\varphi}(v)$$

where $|\cdot|$ is the norm on K such that

$$\text{meas}\,(xE) = |x| \cdot \text{meas}\,(E)$$

for any measurable set E, where meas (E) is the measure of E, and where

$$c = \int_K \psi(\frac{-1}{2}\langle v, v\rangle)\, dv$$

and ν is the norm residue symbol

$$\nu(x) = \begin{cases} 1 & (x \text{ is a norm from } K) \\ -1 & (x \text{ is not a norm from } K) \end{cases}$$

These requirements really are self-consistent and give a representation (see appendix).

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2. *Compatibility condition*

Here we see that ν and c in the definition of the Weil/oscillator representation are necessary for associativity

$$(g \cdot h) \cdot \varphi = g \cdot (h \cdot \varphi)$$

The fragmentary definition of the Weil/oscillator representation on specific group elements does give (as is easy to verify directly) a well-defined representation on the standard parabolic

$$P = \{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \}$$

and a well-defined function on the big Bruhat cell

$$PwN = \{ \begin{bmatrix} * & * \\ \neq 0 & * \end{bmatrix} \}$$

In fact, elementary properties of Fourier transform exactly assure that we have a representation on $M \cup wM$, where M is the standard Levi component

$$M = \{ \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \}$$

The non-trivial compatibility issue is that

$$(g \cdot h)\varphi = g \cdot (h \cdot \varphi)$$

for g, h *both* in the big cell PwN and where the product gh is also in the big cell. Further, since M normalizes N and w normalizes M, a relation

$$n_1 m_1 w n_1' \cdot n_2 m_2 w n_2' = nmwn'$$

(with all n's in N, all m's in M) simplifies to a relation of the form

$$w n_1 w^{-1} = nmwn'$$

where we use $w^{-1} = m_{-1} w$ rather than simply w in light of some computational details that will appear in a moment. Invoking the explicit Bruhat decomposition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = n_{a/c}\, m_{1/c}\, w\, n_{d/c}$$

we have

$$w n_x w^{-1} = \begin{bmatrix} 1 & 0 \\ -x & 1 \end{bmatrix} = n_{-1/x}\, m_{-1/x}\, w\, n_{-1/x}$$

We must prove that the two sides have the same effect on a Schwartz function via the Weil/oscillator representation. On one hand, the left-hand side is

$$w(n_x(w^{-1}\varphi)) = \frac{1}{c} \cdot w(n_x \check{\varphi}) = \frac{1}{c} \cdot w(S_x \check{\varphi}) = \widehat{S_x} * \varphi = \gamma_x \cdot S_{-1/x} * \varphi = \gamma_x \cdot S_{-1/x} \cdot \mu_{-1/x} \mathscr{F}(S_{-1/x}\varphi)$$

using the fact that the function S_x is a tempered distribution, with Fourier transform (see appendix)

$$\widehat{S_x} = \gamma_x \cdot S_{-1/x}$$

where

$$\gamma_x = \int_K \psi(\frac{x}{2}\langle w, w\rangle)\, dw$$

and using the fundamental identity (see appendix)

$$S_x * \varphi = S_x \cdot \mu_x \, \mathscr{F}(S_x \cdot \varphi)$$

On the other hand, the right-hand side is

$$n_{-1/x}(m_{-1/x}(w(n_{-1/x}\varphi))) = n_{-1/x}(m_{-1/x}(w(S_{-1/x}\varphi))) = c \cdot S_{-1/x} \cdot \nu(-1/x)|-1/x|^{1/2}\mu_{-1/x}\mathscr{F}(S_{-1/x}\varphi)$$

These two will be equal if and only if

$$\gamma_x = c \cdot \nu(-1/x)|-1/x|^{1/2}$$

Since x^2 is certainly a norm from any quadratic extension, and since c is defined to be γ_{-1}, this condition is

$$\gamma_x = \gamma_{-1} \cdot \nu(-x)|x|^{-1/2}$$

Proof of the latter uses the fact that norms from $K^\times$ are of index 2 in $k^\times$ for K a field (and index 1 when $K = k \times k$). This index assertion is elementary when K/k is unramified and of residual characteristic not 2, but otherwise is non-trivial.

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3. *Explicit Bruhat decomposition*

As above, let

$$n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \qquad m_x = \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix} \qquad w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,k)$$

with $c \neq 0$, we claim that

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix} = n_{a/c}\, m_{1/c}\, w\, n_{d/c}}$$

This is a formulaic version of the *Bruhat decomposition*. We first *verify* this by multiplying out, and then *derive* it.

$$n_{a/c}\, m_{1/c}\, w\, n_{d/c} = \begin{bmatrix} 1 & a/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/c & a \\ 0 & c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -1/c \\ c & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & \frac{ad}{c} - \frac{1}{c} \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

since $ad - bc = 1$, from the determinant condition. This verifies the relation. On the other hand, once one anticipates that such a relation holds, the precise form of it can be *derived* as follows.

$$\begin{bmatrix} 1 & -a/c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & b - \frac{ad}{c} \\ c & d \end{bmatrix}$$

And

$$\begin{bmatrix} c & 0 \\ 0 & 1/c \end{bmatrix} \begin{bmatrix} 0 & b - \frac{ad}{c} \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & cb - ad \\ 1 & d/c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & d/c \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix} = w\, n_{d/c}$$

That is,

$$m_c\, n_{-a/c} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = w\, n_{d/c}$$

so, indeed,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = n_{a/c}\, m_{1/c}\, w\, n_{d/c}$$

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4. *Fourier transforms and quadratic exponentials*

For non-archimedean k, the tempered distribution $S_x(v) = \psi(\frac{x}{2}\langle v, v\rangle)$ is a weak limit of its truncations to compacts in K. Thus, invoking the continuity of Fourier transform in the weak topology, the integral defining its Fourier transform is the corresponding limit of integrals over compacta. Suppressing this limiting process momentarily, we compute

$$\widehat{S_x}(v) = \int_K \psi(-\langle v, w\rangle)\, S_x(w)\, dw = \int_K \psi(\frac{x}{2}\langle w - \frac{v}{x}, w - \frac{v}{x}\rangle)\, \psi(\frac{-1}{x}\langle v, v\rangle)\, dw = S_{-1/x}(v) \cdot \int_K \psi(\frac{x}{2}\langle w, w\rangle)\, dw$$

by changing variables. The non-archimedean property assures that the change of variables is allowed even in the truncated version of these integrals. As above, for brevity, let

$$\gamma_x = \int_K \psi(\frac{x}{2}\langle w, w\rangle)\, dw$$

so we rewrite the identity as

$$\boxed{\widehat{S_x} = S_{-1/x} \cdot \gamma_x}$$

Further,

$$(S_x * \varphi)(v) = \int_K S_x(v - w)\, \varphi(w)\, dw = \int_K \psi(\frac{x}{2}\langle v - w, v - w\rangle)\, \varphi(w)\, dw$$
$$= S_x(v) \cdot \mathscr{F} S_x \cdot \varphi(vx)$$

That is,

$$\boxed{S_x * \varphi = S_x \cdot \mu_x\, \mathscr{F}(S_x \cdot \varphi)}$$
