(*October 3, 2005*)

Exercises 02

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[2.1] Show that a topological space X is *Hausdorff* if and only if the diagonal $X^\Delta = \{(x,x) \in X \times X : x \in X\}$ is *closed* in $X \times X$.

[2.2] The usual notion of *quotient* of a topological space X by an equivalence relation $\sim$ on X may product non-Hausdorff spaces. Show that for the quotient $X/\sim$ to be Hausdorff it is *necessary* that the $\sim$-equivalence classes be *closed* subsets of X.

[2.3] Show that for a topological quotient $X/\sim$ (of a Hausdorff space X) to be Hausdorff it is *sufficient* that the $\sim$-equivalence classes be *compact*. (Give an example to see that this condition is certainly not *necessary*.)

[2.4] Find the colimit (as abelian groups) of

$$\mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 2} \cdots$$

[2.5] Find the colimit (as abelian groups) of

$$\mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\times 3} \mathbf{Z} \xrightarrow{\times 5} \mathbf{Z} \xrightarrow{\times 7} \mathbf{Z} \xrightarrow{\times 11} \mathbf{Z} \xrightarrow{\times 13} \cdots$$

[2.6] Find the colimit of

$$\mathbf{R}/\mathbf{Z} \xrightarrow{\bmod \frac{1}{2}} \mathbf{R}/\tfrac{1}{2}\mathbf{Z} \xrightarrow{\bmod \frac{1}{4}} \mathbf{R}/\tfrac{1}{4}\mathbf{Z} \xrightarrow{\bmod \frac{1}{8}} \cdots$$

[2.7] In the category of *Hausdorff* topological spaces and continuous maps among them, and given an equivalence relation $\sim$ on a Hausdorff space X, the **Hausdorff quotient** Q of X by $\sim$ is, by definition (if it exists), a Hausdorff topological space Q with a surjective map $q : X \longrightarrow Q$ with Q given the quotient topology (a set U in Q is open if and only if $q^{-1}(U)$ is open in X), and such that any continuous map $f : X \longrightarrow Y$ to a Hausdorff space Y constant on $\sim$-equivalence classes *(uniquely) factors through* q. Do Hausdorff quotients exist?

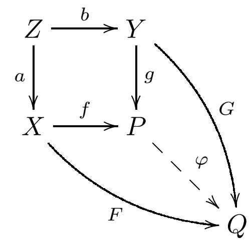
[2.8] Given objects X, Y, Z and maps $a : Z \longrightarrow X$ and $b : Z \longrightarrow Y$, a **pushout** is an object P with maps $f : X \longrightarrow P$ and $g : Y \longrightarrow P$ giving a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{b} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & P \end{array}$$

and such that, for every object Q and maps $F : X \longrightarrow Q$ and $G : Y \longrightarrow Q$ giving a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{b} & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle G} \\ X & \xrightarrow{F} & Q \end{array}$$

there is a unique $\varphi : P \longrightarrow Q$ giving a commutative diagram



Note that the case that for $Z = \{1\}$ is the *coproduct* of X and Y. For not-necessarily abelian groups, for example, prove that pushouts exist.

[2.9] Reversing the arrows for the pushout of two objects, define the *pullback* of two objects. Do all pullbacks of groups exist?