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Exercises 03

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Definition: Let $\mathrm{Hom}(A, B)$ denote the *set* of all maps from object A to object B.

[3.1] Show that a finite set has a *unique* Hausdorff topology.

[3.2] Give an example of a non-Hausdorff space with a compact subset that is *not* closed.

[3.3] Prove that a colimit of coproducts is the coproduct of the colimits.

[3.4] Let $X = \lim_i X_i$ be a limit of finite sets X. Let $p_i : X \longrightarrow X_i$ be the projection to the i^{th} limitand. Show that the topology on X has a *basis* of sets $p_i^{-1}(x_i)$ for points $x_i \in X_i$.

[3.5] Map $\mathbf{Z} \longrightarrow \mathbf{Z}/n$ by $z \longrightarrow z + n\mathbf{Z}$ as usual. Show that $\mathbf{Z}$ is *dense* in the limit $\widehat{Z} = \lim_n \mathbf{Z}/n$ under the induced map. Show that the sequence of factorials $n!$ goes to 0 in $\widehat{Z}$.

[3.6] Let $\{X_i : i \in I\}$ be objects indexed by a set I. Show that there is a natural isomorphism

$$\mathrm{Hom}(\coprod_i X_i, Y) \approx \prod_i \mathrm{Hom}(X_i, Y)$$

and

$$\mathrm{Hom}(Y, \prod_i X_i) \approx \prod_i \mathrm{Hom}(Y, X_i)$$

[3.7] Let $\{X_i : i \in I\}$ be objects indexed by a poset I. Show that there is a natural isomorphism

$$\mathrm{Hom}(\mathrm{colim}_i X_i, Y) \approx \lim_i \mathrm{Hom}(X_i, Y)$$

and

$$\mathrm{Hom}(Y, \lim_i X_i) \approx \lim_i \mathrm{Hom}(Y, X_i)$$

[3.8] Let μ_n be the set of all n^{th} roots of unity inside a fixed algebraic closure $\overline{\mathbf{Q}}$ of the rationals $\mathbf{Q}$. Let Ω be the field obtained by adjoining all μ_n to $\mathbf{Q}$, and let

$$\mathrm{Gal}(\Omega/\mathbf{Q}) = \lim_n \mathrm{Gal}(\mathbf{Q}(\mu_n)/\mathbf{Q})$$

where the indices n are ordered by divisibility. Prove that

$$\mathrm{Gal}(\Omega/\mathbf{Q}) = \lim_n (\mathbf{Z}/n)^\times$$

[3.9] Let $\mathbf{F}_q$ denote a finite field with q elements. Show that the Galois groups of the (separable) algebraic closure of $\mathbf{F}_q$ is naturally isomorphic to $\widehat{Z} = \lim_n \mathbf{Z}/n$, where the indices are ordered by divisibility.

[3.10] *(Infinite Galois theory)* Let k be a field. Let I be the poset of finite Galois extensions K of k, ordered by inclusion. Define the Galois group of the (separable) algebraic closure $\overline{k}$ of k by

$$\mathrm{Gal}(\overline{k}/k) = \lim_{K \in I} \mathrm{Gal}(K/k)$$

as a *topological* group. Show that the finite (separable) extensions of k are in bijection with *open* subgroups H of G, by

$$H \leftrightarrow \text{fixed field of } H$$

Show that the not-necessarily finite (separable algebraic) extensions of k are in bijection with *closed* subgroups of G.