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Modular forms and number theory solution/discussion 01

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The question was:

[mfms 01.1] Prove the Euler product expansion of the zeta function, namely, for $\mathrm{Re}(s) > 1$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$$

using the geometric series expansion

$$\frac{1}{1-p^{-s}} \;=\; 1 + p^{-s} + (p^2)^{-s} + (p^3)^{-s} + \ldots$$

Often this Euler product expansion is interpreted as a slightly analytic manifestation of the *unique factorization* in $\mathbb{Z}$.

[0.1] The main issue One central point is the discrepancy between *finite* products of *finite* geometric series involving primes, and *finite* sums of natural numbers. For example, for $T > 1$, because every positive integer $n < T$ is a product of prime powers $p^m < T$ in a unique manner,

$$\Big|\prod_{\text{prime } p<T} \Big(\sum_{m\,:\,p^m<T} \frac{1}{p^{ms}}\Big) \;-\; \sum_{n<T} \frac{1}{n^s}\Big| \;<\; \sum_{n\geq T} \frac{1}{|n^s|}$$

since the finitely-many leftovers from the product produce integers $n \geq T$ at most once each. The latter sum goes to 0 as $T \to \infty$, for fixed $\mathrm{Re}(s) > 1$, by comparison with an integral.

The finite sums $\sum_{n<T} 1/n^s$ are the usual partial sums of the infinite sum, and (simple) convergence is the assertion that this sequence converges to $\sum_n 1/n^s$.

Thus, this already proves that

$$\lim_{T\to\infty} \prod_{\text{prime } p<T} \Big(\sum_{m\,:\,p^m<T} \frac{1}{p^{ms}}\Big) \;=\; \sum_{n\geq 1} \frac{1}{n^s}$$

[0.2] Limits of varying products In contrast, the auxiliary question about infinite products is more complicated. We have products *whose factors themselves vary*: we would like to prove that because

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots + \frac{1}{p^{ms}} \;\longrightarrow\; \frac{1}{1-\frac{1}{p^s}} \qquad \text{(for } \mathrm{Re}(s) > 1\text{)}$$

the limit of *changing* products converges:

$$\prod_{p<T} \Big(\sum_{m:p^m<T} \frac{1}{p^{ms}}\Big) \;\longrightarrow\; \prod_p \frac{1}{1-\frac{1}{p^s}} \qquad \text{(for } \mathrm{Re}(s) > 1\text{)}$$

where the infinite product on the right is the limit of its finite partial products. *That the individual factors on the left approach the individual factors on the right is not necessary (for fixed s), and in any case is not sufficient.*

Taking logarithms is convenient, since error estimates on *sums* is easier than error estimates on *products*. That is, we claim that

$$\log \prod_{p<T} \Big(\sum_{m:p^m<T} \frac{1}{p^{ms}} \Big) \longrightarrow \log \prod_p \frac{1}{1-\frac{1}{p^s}} \qquad \text{(for Re}(s) > 1)$$

The infinite product on the right is easily verified to converge to a non-zero limit, so continuity of *log* away from 0 allows us to move the logarithm inside the infinite product. Moving *log* inside a *finite* product is not an issue. Thus, it suffices to prove that

$$\sum_{p<T} \log \Big(\sum_{m:p^m<T} \frac{1}{p^{ms}} \Big) \longrightarrow \sum_p \log \frac{1}{1-\frac{1}{p^s}} \qquad \text{(for Re}(s) > 1)$$

[0.2.1] Claim: For fixed $0 < \delta < 1$, there is a constant $C > 0$ such that, for any $|x| < \delta$ and $|y| < \delta$,

$$\Big| \log(1+x) - \log(1+y) \Big| < C \cdot |x-y|$$

(This is the mean value theorem.) ///

Now the approximations of the factors by geometric series can be used: for fixed p, letting $\sigma = \text{Re}(s) > 1$,

$$\Big| \sum_{m:p^m<T} \frac{1}{p^{ms}} - \frac{1}{1-\frac{1}{p^s}} \Big| \le \sum_{m:p^m \ge T} \frac{1}{p^{m\sigma}} \le \frac{1}{T^\sigma} \cdot \frac{1}{1-2^{-\sigma}}$$

Thus, for fixed $\sigma > 1$, for every p

$$\Big| \log \sum_{m:p^m<T} \frac{1}{p^{ms}} - \log \frac{1}{1-\frac{1}{p^s}} \Big| \le C \cdot \frac{1}{T^\sigma} \cdot \frac{1}{1-2^{-\sigma}}$$

and then

$$\sum_{p<T} \Big| \log \sum_{m:p^m<T} \frac{1}{p^{ms}} - \log \frac{1}{1-\frac{1}{p^s}} \Big| < \frac{1}{T^\sigma} \sum_{p<T} \frac{1}{1-2^{-\sigma}} \le \frac{1}{T^{\sigma-1}} \cdot \frac{1}{1-2^{-\sigma}}$$

Given $\varepsilon > 0$, take T_o large enough so that for $T > T_o$

$$\Big| \sum_{p<T} \log \frac{1}{1-\frac{1}{p^s}} - \sum_p \log \frac{1}{1-\frac{1}{p^s}} \Big| < \varepsilon$$

Then

$$\Big| \sum_{p<T} \log \Big(\sum_{m:p^m<T} \frac{1}{p^{ms}} \Big) - \sum_p \log \frac{1}{1-\frac{1}{p^s}} \Big| \le \Big| \sum_{p<T} \log \Big(\sum_{m:p^m<T} \frac{1}{p^{ms}} \Big) - \sum_{p<T} \log \frac{1}{1-\frac{1}{p^s}} \Big| + \varepsilon$$

$$< \frac{C}{1-2^\sigma} \cdot \frac{1}{T^{\sigma-1}} + \varepsilon$$

Since $\sigma > 1$, this can be made small by increasing T. ///

[0.2.2] Remark: Since everything turned out nicely, one might think that the above discussion made too much of a fuss about small issues. Indeed, nothing counter-intuitive transpired. However, at the least, it is worthwhile to understand exactly what the assertion of an Euler product factorization entails, in terms of *finite* expressions.

[0.3] Discussion One issue is a sensible meaning for *convergence* of an infinite *product* $\prod_{i=1}^\infty a_i$. Our general understanding of *infinite* processes rests on essentially a single notion, that of taking a *limit* of finite subprocesses. An *ordering* may be further specified, to distinguish a special class of finite subprocesses. For example, an infinite sum $\sum_{i=1}^\infty a_i$ has value the limit, if that limit exists, of the *special* finite subsums $s_N = \sum_{i=1}^N a_i$. We could make the stronger requirement of convergence of the *net* [1] of *all* finite subsums, indexed by the *directed poset* of *all* finite subsets of $\{1, 2, \ldots\}$. Convergence of such a more complicated net would mean that, given $\varepsilon > 0$, there is a finite subset $F \subset \{1, 2, \ldots\}$ such that, for any finite subsets X, Y of $\{1, 2, \ldots\}$ containing F,

$$\Big|\sum_{i\in X} a_i \;-\; \sum_{i\in Y} a_i\Big| \;<\; \varepsilon$$

One can prove that this stronger notion of convergence is equivalent to *absolute* convergence. For subsequent manipulations of infinite sums, usually we want and need absolute convergence.

Similarly, for infinite products $\prod_{i=1}^\infty a_i$, the weakest reasonable convergence requirement is convergence of the *sequence* of finite sub-products $\prod_{i=1}^N a_i$. *Absolute* convergence is equivalent to the stronger requirement of convergence of net of *all* finite sub-products. The stronger requirement is necessary to legitimize non-trivial subsequent manipulations.

The behavior of 0 in multiplication has an effect on infinite products with no counterpart in infinite sums, namely, that a single factor of 0 makes the whole product 0, regardless of the behavior of other factors. This might seem silly or undesirable, so some sources declare this behavior unacceptable, or given an impression of compromise by allowing only *finitely-many* factors of 0.

A more serious issue is *convergence to* 0. For example, the sequence of finite partial products

$$p_T \;=\; \prod_{n\le T} \big(1 - \frac{1}{n}\big)$$

converges to 0. Thus, it makes sense to say that the *infinite* product converges to 0. However, many sources disallow this, as part of their definition. On the face of it, there is no reason to object to convergence to 0, since it certainly fits with general principles about infinite processes being limits of finite sub-processes.

However, in the present situation as well as in many other applications, one immediately takes a *logarithm* of a product. Thus, we want infinite products which converge in a sense that makes the infinite sum of logarithms converge. The discrepancy between this goal and the general principles about infinite processes being limits of finite sub-processes is genuine, since logarithm is not *continuous* at 0. *It would be unreasonable to expect maps such as log to preserve limits at points where they are not continuous.*

For example, taking logarithms in the product displayed above,

$$\sum_n \log\big(1 - \frac{1}{n}\big) \;=\; \sum_n -\big(\frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots\big) \;\le\; -\sum_n \frac{1}{n} \;=\; -\infty$$

That is, as expected, convergence of an infinite product to 0 becomes divergence to $-\infty$ under logarithm.

[1] A *net* is a useful generalization of *sequence*: while a sequence is a set indexed by the ordered set $\{1, 2, \ldots\}$, a *net* is a set indexed by a *directed poset*. The word *poset* is a common abbreviation for *partially ordered set*, which is a set with a partial order $x < y$. A partial order is *transitive*, meaning that $x < y$ and $y < z$ implies $x < z$, and anti-symmetric, meaning that $x \not< x$. The *directed* condition on a poset S is that, given $x, y \in S$, there is $z \in S$ with both $x < z$ and $y < z$.

A detail: logarithms of infinite products of *complex* numbers require conventions to avoid meaningless divergence due to ambiguities in the imaginary part of logarithms. This issue is secondary, so we ignore it in the present discussion.

In any context in which logarithms of products matter, we might *define* convergence of a product $\prod_j a_j$ of positive reals a_j to be convergence of the sum $\sum_j \log a_j$, and expect to prove

[0.3.1] Claim: For positive real numbers a_j, if the infinite sum $\sum_j \log a_j$ converges, then $\prod_j a_j$ converges, in the sense that the sequence of partial products $p_N = \prod_{j \le N} a_j$ converges. ///

In terms of logarithms, from

$$-\log(1-x) \;=\; x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \qquad \text{(for } |x| < 1\text{)}$$

we have approximations that simplify sums of logarithms. For example, for $\delta > 0$, there are $A, B > 0$ such that

$$Ax \;<\; -\log(1-x) \;<\; Bx \qquad \text{(for } |x| < 1-\delta\text{, constants } A, B \text{ depending on } \delta > 0\text{)}$$

Thus, an infinite product $\prod_n (1 + a_n)$ with all the a_n's in the range $|a_n| < 1 - \delta$ has partial products with logarithms satisfying

$$A \sum_{n<N} a_n \;<\; \log \prod_{n<N} (1 + a_n) \;=\; \sum_{n<N} \log(1 + a_n) \;<\; B \sum_{n<N} a_n \qquad (A, B \text{ depending on } \delta > 0)$$

Thus, in this situation, convergence of the sum of logarithms $\log(1 + a_n)$ is equivalent to convergence of the sum of a_n. There are obvious variations.

Again, infinite products may converge to 0 in the sense that the sequence of partial subproducts converges to 0, but the sequence of sums of logarithms of partial subproducts diverges to $-\infty$. Equivalently, the comparison of $\log(1 + x)$ and x fails as $x \to -1$. Equivalently, log is not continuous at 0.