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# *Modular forms and number theory exercises 02*

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Starred exercises are for more advanced students, though they're by no means unreasonable!

**[mfms 02.1]** Verify an extended variant of **Perron's identity**: compute by residues that, for $0 < \ell \in \mathbb{Z}$, $\theta > 0$, and $\sigma > 0$,

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{sX}\ ds}{s(s+\theta)(s+2\theta)\ldots(s+\ell\theta)} \;=\; \begin{cases} \dfrac{1}{\ell!\,\theta^\ell} \cdot \big(1 - e^{-X\theta}\big)^\ell & (\text{for } X > 0) \\ 0 & (\text{for } X < 0) \end{cases}$$

**[mfms 02.2]** (*) Prove that the $8^{th}$ **cyclotomic polynomial** $x^4 + 1$ is *reducible* as a polynomial in $\mathbb{F}_p[x]$ for *every* odd prime $p$, where $\mathbb{F}_p$ is the finite field with $p$ elements. Note that $x^4 + 1$ is *irreducible* in $\mathbb{Q}[x]$ or $\mathbb{Z}[x]$, by Eisenstein's criterion applied to $(x+1)^4 + 1$.

**[mfms 02.3]** (*) Similarly, for distinct odd primes $p, q$, show that, although the $pq^{th}$ cyclotomic polynomial

$$\varphi(x) \;=\; \frac{x^{pq} - 1}{(x-1)\,(x^{p-1} + x^{p-2} + \ldots + x + 1)\,(x^{q-1} + x^{q-2} + \ldots + x + 1)} \;=\; \frac{(x^{pq} - 1)\,(x-1)}{(x^p - 1)\,(x^q - 1)}$$

is *irreducible* in $\mathbb{Q}[x]$ or $\mathbb{Z}[x]$, it is *reducible* in $\mathbb{F}_r[x]$ for any prime $r$.