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Modular forms and number theory exercises 03

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[mfms 03.1] (a)Show that there is a unique sequence of polynomials $B_1(x), B_2(x), B_3(x), \ldots$ such that

$$\int_0^1 B_\ell(x)\,dx \;=\; 0 \qquad \text{and} \qquad \int_0^1 B_\ell(x)\,e^{-2\pi i n x}\,dx \;=\; \frac{-1}{(2\pi i n)^\ell} \qquad (\text{for all } 0 \neq \ell \in \mathbb{Z})$$

(b) Determine the first few $B_\ell(x)$ explicitly, and use them to evaluate

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots$$

$$\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \ldots$$

$$\frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \ldots$$

(c) Determine the Euler factorization of

$$L(s) \;=\; \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \ldots$$

and obtain $\zeta(2), \zeta(4), \zeta(6)$ as corollaries.

[0.0.1] Remark: The polynomials $B_\ell(x)$ are essentially **Bernoulli polynomials**.

[0.0.2] Remark: To get zeta values, another similar approach is invocation of a **Plancherel theorem** [1] that for a Fourier expansion $f(x) \sim \sum_n a_n e^{2\pi i n x}$,

$$\int_0^1 |f(x)|^2\,dx \;=\; \sum_n |a_n|^2$$

In this approach, only $B_1(x), B_2(x), B_3(x)$ are needed, but we must integrate their squares. In the approach sketched above, we need $B_2(x), B_4(x), B_6(x)$, but no squaring and no subsequent integration is needed.

[0.0.3] Remark: For hand computations with Bernoulli polynomials, it's convenient to write them as polynomials in $x - \frac{1}{2}$. For example, if $B_\ell(x)$ is described as the integral of $B_{\ell-1}(x)$ plus a constant (to meet the $\int_0^1 B_\ell(x)\,dx = 0$ condition), then the constant required in going from even index to odd index is simply 0. Further, for both parities, that constant becomes the value $B_\ell(\frac{1}{2})$.

[mfms 03.2] (*) Find a small, closed-form expression for the *generating function* [2] $\sum_\ell t^\ell\, B_\ell(x)$.

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[1] Proof of the Plancherel theorem depends upon first results on Hilbert spaces. This is a very mild prerequisite, easy to meet, as will be done shortly. The notion of *Plancherel theorem* makes sense in many, many situations, so is a device one can hope to rely upon systematically.

[2] As usual, a *generating function* for a list $b_1, b_2, \ldots$ of numbers of other reasonable objects is an *a priori* infinite sum $\sum_n b_n t^n$ or $\sum_n b_n t^n/n!$ or other form, so that this infinite sum happens to have a convenient *closed form*, meaning a significantly simpler or more elementary expression. There are no guarantees of existence of simpler closed forms, and choice of details of a proposed infinite sum is an art in itself.