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Modular forms and number theory exercises 09

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[mfms 09.1] Let $z = x + iy$, $\overline{z} = x - iy$, and

$$\frac{\partial}{\partial z} = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \qquad\qquad \frac{\partial}{\partial \overline{z}} = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

Prove that the obviously-suggested identities hold:

$$\frac{\partial}{\partial z}(z^m\,\overline{z}^n) = m\,z^{m-1}\,\overline{z}^n \qquad\qquad \frac{\partial}{\partial \overline{z}}(z^m\,\overline{z}^n) = n\,z^m\,\overline{z}^{n-1}$$

[mfms 09.2] For real θ, let $\mu = e^{i\theta}$, so $|\mu| = 1$. Let

$$u(\mu) = \sum_{n\in\mathbb{Z}} c_n \mu^n = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

be the Fourier expansion of a *smooth* function u. Suppose that $u(-\mu) = u(\mu)$. Show that $c_n = 0$ unless $2|n$. Define *grossencharacters* on $\mathbb{Q}(i)$ by

$$\chi_\ell(z) = \big(z/\overline{z}\big)^\ell \qquad\qquad (\text{with } \ell \in \mathbb{Z})$$

and grossencharacter L-function for the Gaussian integers

$$L(s,\chi_\ell) = \frac{1}{4}\sum_{0\neq\alpha\in\mathbb{Z}[i]} \frac{\chi_\ell(\alpha)}{(\alpha\overline{\alpha})^s} = \prod_\pi \frac{1}{1-\dfrac{\chi_\ell(\pi)}{(\pi\overline{\pi})^s}} \qquad\qquad (\pi \text{ runs over Gaussian primes})$$

Show that

$$\sum_\ell c_{2\ell}\log L(s,\chi_\ell) = \sum_\pi \frac{u(\pi/\overline{\pi})}{|\pi\overline{\pi}|^s} + (\text{holomorphic at } s=1) \qquad\qquad (\pi \text{ runs over Gaussian primes})$$

Convergence?

[mfms 09.3] (*) Prove that $u(z) = 1/z$ is locally integrable on $\mathbb{C}$, with respect to the usual measure. Thus, it can be treated as a *distribution*, by

$$u(f) = \int_{\mathbb{C}} \frac{f(z)}{z}\,dx\,dy \qquad\qquad (\text{for a } \textit{test function } f)$$

The function $1/z$ is holomorphic away from 0, so is annihilated by the Cauchy-Riemann operator $\partial/\partial\overline{z}$ away from 0. Its behavior at 0 is subtler: prove that

$$\frac{\partial}{\partial\overline{z}}\frac{1}{z} = \pi\cdot\delta \qquad\qquad (\text{where } \delta \text{ is Dirac delta})$$

(*Hint:* One approach is to consider the holomorphically parametrized family $u_s(z) = \frac{1}{z}\cdot|z\overline{z}|^s$ of distributions, with complex s with $\mathrm{Re}(s) > 0$, and look at the residue of the first pole, namely, as $s \to 0^+$. Observe that there are some difficulties in using Fourier transform methods here.)