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Modular forms and number theory exercises 10

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[mfms 10.1] A finite version of a *local functional equation*. Fix a prime p. For a multiplicative character $\chi : (\mathbb{Z}/p)^\times \to \mathbb{C}^\times$ and an arbitrary function f on $\mathbb{Z}/p$, define a **zeta function**

$$Z(\chi, f) \;=\; \sum_{x \in \mathbb{Z}/p} \chi(x)\, f(x)$$

Fix a non-trivial additive character $\psi : \mathbb{Z}/p \to \mathbb{C}^\times$, and define a Fourier transform

$$\widehat{f}(\xi) \;=\; \sum_{x \in \mathbb{Z}/p} f(x)\, \psi(x\xi)$$

For two functions f, g on $\mathbb{Z}/p$, prove the *local functional equation*

$$Z(\chi, f)\, Z(\overline{\chi}, \widehat{g}) \;=\; Z(\overline{\chi}, \widehat{f})\, Z(\chi, g)$$

[mfms 10.2] (*) Let G be an abelian *totally disconnected* topological group. That is, there is a local basis at $e \in G$ consisting of *subgroups*. Prove that G is homeomorphic to the (projective) limit of the quotients G/K as K runs over neighborhoods of e.

[mfms 10.3] (*) Prove that the unitary dual of $\mathbb{Q}_p$ is isomorphic to $\mathbb{Q}_p$ (non-canonically, however). Specifically, given a non-trivial additive character ψ on $\mathbb{Q}_p$ prove that *every* additive character on $\mathbb{Q}_p$ is of the form $x \to \psi(\xi x)$ for a unique $\xi \in \mathbb{Q}_p$.

[mfms 10.4] (*) A $\mathbb{C}$-valued function f on $\mathbb{Q}_p$ is *locally constant* when, for every $x \in \mathbb{Q}_p$, there is an open subgroup U_x of $\mathbb{Q}_p$ such that

$$f(x+u) \;=\; f(x) \qquad \text{(for all } u \in U_x)$$

A function f is *uniformly* locally constant when a single open U fulfills that requirement for *every* x. Prove that a *compactly-supported* locally constant function is *uniformly* locally constant. Define the *Schwartz-Bruhat space* $\mathscr{S}(\mathbb{Q}_p)$ to be the collection of $\mathbb{C}$-valued, locally-constant, compactly supported functions on $\mathbb{Q}_p$. Define an *integral* of $f \in \mathscr{S}(\mathbb{Q}_p)$ as follows: for sufficiently small open subgroup U of $\mathbb{Z}_p \subset \mathbb{Q}_p$ so that f is invariant under U, show that the support of f is a disjoint union of cosets $x_i + U$ for a finite set x_i. Define

$$\int_{\mathbb{Q}_p} f \;=\; \frac{1}{[\mathbb{Z}_p : U]} \sum_i f(x_i)$$

Show that this is independent of sufficiently small U.

[mfms 10.5] (*) For $f \in \mathscr{S}(\mathbb{Q}_p)$, for a fixed non-trivial additive character ψ on $\mathbb{Q}_p$, and for fixed $\xi \in \mathbb{Q}_p$, show that $x \to f(x)\,\overline{\psi}(x\xi)$ is in $\mathscr{S}(\mathbb{Q}_p)$. Define the **Fourier transform** $f \to \widehat{f}$ by

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{Q}_p} f(x)\, \overline{\psi}(x\xi)\, dx$$

Show that $\widehat{f} \in \mathscr{S}(\mathbb{Q}_p)$. Prove Fourier inversion: $\widehat{\widehat{f}}(x) = f(-x)$.