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Modular forms and number theory exercises 13

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[mfms 13.1] Show that the matrix exponential series

$$e^M = \sum_{\ell=0}^{\infty} \frac{M^\ell}{\ell!} \qquad \text{(for } n\text{-by-}n \text{ complex matrix } M\text{)}$$

converges absolutely. To make sense of the claim, one must have a topology. In fact, there is a unique reasonable (Hausdorff, translation-invariant, etc.) topology on the space of matrices of a given size. Thus, all apparent descriptions of reasonable topology give the same thing. Granting that, there is certainly no harm in making the convenient choice of a metric topology coming from the (uniform operator) *norm*

$$\|M\| = \sup_{x \in \mathbb{C}^n:\ |x| \le 1} |Mx|$$

where $|x|^2 = |x_1|^2 + \ldots + |x_n|^2$ is the usual hermitian norm. Then the metric on matrices is

$$d(M, N) = \|M - N\|$$

One might verify that this truly is a metric, and then prove convergence of the exponential series.

[mfms 13.2]* Prove that the so-called *trace form*

$$\langle x, y \rangle = \operatorname{tr}(xy) \qquad \text{(with matrix multiplication and matrix trace)}$$

on the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ is a scalar multiple of the Killing form, by showing that the space of $\mathfrak{g}$-invariant vectors in $\mathfrak{g} \otimes \mathfrak{g}$ is one-dimensional. Determine the multiple.