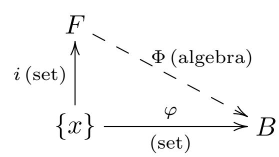
(*March 12, 2011*)

Modular forms and number theory exercises 15

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[mfms 15.1] Let $\mathfrak{g}$ be a one-dimensional Lie algebra, with a basis $\{x\}$, and $[x,x]=0$. Show that both the universal associative algebra $A\mathfrak{g}$ and the universal enveloping algebra $U\mathfrak{g}$ are isomorphic to a polynomial ring in one variable.

Hint: it may be helpful to characterize the polynomial ring $k[x]$ in one variable over a commutative ring k with 1 as being the *free k-algebra F on the set* $\{x\}$, in the following sense. There is a fixed set map $i:\{x\}\to F$ such that, given a *set map* $\varphi:\{x\}\to B$ to a commutative k-algebra B, there is a unique commutative k-algebra homomorphism $\Phi:F\to B$ giving a commutative diagram



[mfms 15.2] The point of this (protracted) exercise is to compute the Casimir operator in the *unit disk* model of hyperbolic two-space (inside $\mathbb{C}$), with the *special unitary group*

$$G \;=\; SU(1,1) \;=\; \{g\in SL_2(\mathbb{C})\;:\; g^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}g=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\} \qquad (g^* \text{ is conjugate-transpose})$$

acting by linear fractional transformations as usual:

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}(z) \;=\; \frac{az+b}{cz+d}$$

This is a different-coordinates version of the upper half-plane $\mathfrak{H}$ with $SL_2(\mathbb{R})$ acting. Let's not worry about checking that the group really does stabilize the unit disk $\mathfrak{D}$. Let

$$S \;=\; \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

First, show that the Lie algebra is

$$\mathfrak{g} \;=\; \mathfrak{su}(1,1) \;=\; \{\text{complex 2-by-2 }\gamma\;:\;\gamma^*S+S\gamma=0 \text{ and } \mathrm{tr}(\gamma)=0\}$$

$$=\;\{\begin{pmatrix}ia&\beta\\\overline{\beta}&-ia\end{pmatrix}\;:\;a\in\mathbb{R},\;\beta\in\mathbb{C}\}$$

Second, show that

$$\langle x,y\rangle \;=\; \mathrm{Re}\big(\mathrm{tr}(xy)\big) \qquad (\mathrm{Re}\,(z) \text{ is } \textit{real part} \text{ of complex } z)$$

is a G-conjugation-invariant non-degenerate real-valued $\langle,\rangle$ on $\mathfrak{g}$. Third, show that

$$h \;=\; \begin{pmatrix}i&0\\0&-i\end{pmatrix} \qquad u=\begin{pmatrix}0&1\\1&0\end{pmatrix} \qquad v=\begin{pmatrix}0&i\\-i&0\end{pmatrix}$$

form an *orthogonal* basis for $\mathfrak{g}$, with

$$\langle h,h\rangle \;=\; -2 \qquad \langle u,u\rangle \;=\; 2 \qquad \langle v,v\rangle \;=\; 2$$

Note the signs. Thus, up to a constant, the Casimir element is

$$\Omega \;=\; -h^2 + u^2 + v^2 \;\in\; U\mathfrak{g}$$

Determine $\exp(th)$, $\exp(tu)$, $\exp(tv)$. Next, since Casimir involves second derivatives and nothing higher, express each of the images $\exp(th)(z)$, $\exp(tu)(z)$, $\exp(tv)(z)$ for $z \in \mathfrak{D}$ modulo t^3, meaning in the form

$$A(z) + tB(z) + t^2C(z) \qquad\qquad \text{(where } A, B, C \text{ are simple functions of } z\text{)}$$

As an example, mod t^3, keeping in mind $(1-a)^{-1} = 1 + a + a^2 + \ldots$,

$$\exp(tu)(z) \;=\; \begin{pmatrix} 1+\frac{t^2}{2} & t \\ t & 1+\frac{t^2}{2} \end{pmatrix}(z) \;=\; \frac{(1+\frac{t^2}{2})z + t}{tz + (1+\frac{t^2}{2})} \;=\; \Big(z + t + \frac{t^2}{2}z\Big)\Big(1 + tz + \frac{t^2}{2}\Big)^{-1}$$

$$= \Big(z + t + \frac{t^2}{2}z\Big)\Big(1 - \big(tz + \frac{t^2}{2}\big) + \big(tz + \frac{t^2}{2}\big)^2\Big) \;=\; \Big(z + t + \frac{t^2}{2}z\Big)\Big(1 - tz - \frac{t^2}{2} + t^2z^2\Big)$$

$$= z + t + \frac{t^2}{2}z - tz^2 - t^2z - \frac{t^2}{2}z + t^2z^3 \;=\; z + t(1-z^2) + t^2(-z+z^3)$$

Finally, use the latter expressions and the chain rule to compute

$$\Omega F(x,y) \qquad\qquad \text{(for } z = x+iy \in \mathfrak{D}\text{)}$$

Note: the $-h^2$ term should be easy. In light of the previous computation, and continuing to compute modulo t^3, the u^2 term is

$$\frac{\partial^2}{\partial t^2}\Big|_{t=0} F\big(\exp(-tu)(z)\big) \;=\; \frac{\partial^2}{\partial t^2}\Big|_{t=0} F\big(\exp(tu)(z)\big) \;=\; \frac{\partial^2}{\partial t^2}\Big|_{t=0} F\big(z + t(1-z^2) + t^2(-z+z^3)\big)$$

$$= \frac{\partial}{\partial t}\Big|_{t=0} \Big(\big((1-z^2) + 2t(-z+z^3)\big)\cdot F_z\big(z + t(1-z^2)\big)\Big)$$

$$= 2(-z+z^3)F_z(z) + (1-z^2)(1-z^2)F_{zz}(z)$$

That is, as differential operator on $\mathfrak{D}$,

$$u^2 \;=\; -2z(1-z^2)\frac{\partial}{\partial z} + (1-z^2)^2\frac{\partial^2}{\partial z^2}$$

Carry out the same computation for v^2.

[mfms 15.3]* Show that the special unitary group $SU(1,1)$ is also expressible as

$$SU(1,1) \;=\; \{\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \;:\; \alpha,\beta \in \mathbb{C} \text{ with } |\alpha|^2 - |\beta|^2 = 1\}$$

[mfms 15.4]* Justify the computation above in which t^3 and higher terms are dropped.

[mfms 15.5]* Give an example, with proof, of a real 2-by-2 matrix *with determinant* 1 which is not hit by the exponential map from real 2-by-2 matrices.