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Example of characterization by mapping properties: the product topology

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To communicate clearly in mathematical writing, it is helpful to clearly express *intentions*, as opposed to coyly constructing things whose purpose becomes clear only later.

Often it is not the *internal structure* of a thing that is interesting, but its *interactions* with *other objects*. That is, often we have little long-term interest in the details of the *construction* of the thing, but care more about how it *behaves*. Thus, to express our genuine intentions, we should *not* first *construct* the thing, and only gradually admit that it does what we had planned all along. Instead, we should tell what *external interactions* we demand or expect, and worry about internal details later.

(Admittedly, at earlier stages in one's mathematical development, this style might have been unhelpful. Conceding this, we still do want to warn against accidentally getting stuck in developmental stages that are obsolete.)

Surprisingly, often the characterizations of an object in terms of maps to and from other objects of the same sort succeed in uniquely determining the thing. Even more surprisingly, often this uniqueness follows merely from the shape of the diagrams of the maps, not from any subtler features of the maps or objects.

As practice in using mapping-property characterizations, we surely should first reconsider familiar objects in this light, before trying this approach in unfamiliar circumstances.

For example, the nature of the *product topology* on products of topological spaces [1] is illuminated by this approach. In particular, one might have (at some point) wondered *why the product topology is so coarse.* That is, on infinite products the product topology is strictly coarser than the *box topology.* [2] The answer is that the question itself is misguided, since *the product topology is what it has to be.* That is, there is no genuine *choice* in the construction. Of course, this sort of answer itself needs explanation.

[1] Recall that a *topological space* is a set X with a specified collection of subsets, called *open sets*, such that ϕ and X are open, finite intersections of opens are open, and arbitrary unions of opens are open. We will not need or use any subtler features than essentially this definition.

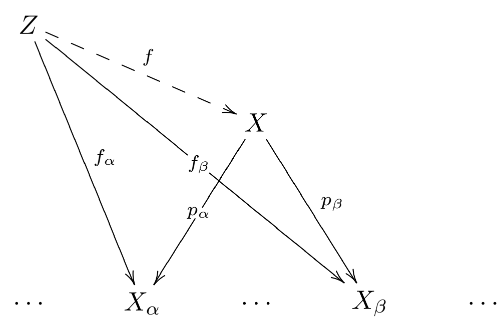
[2] Recall that the *product topology* on the Cartesian product $\prod_{\alpha \in A} X_\alpha$ of topological spaces X_α is usually defined to be the topology with basis consisting of sets $\prod_{\alpha \in A} U_\alpha$ with U_α open in X_α, and only finitely-many of the U_α's different from X_α. The latter constraint struck me as disappointing. Dropping that condition gives the *box topology*, so named for the obvious reason.

1. *Definition and uniqueness of products*

A **product** of non-empty topological spaces X_α for α in an index set A is a topological space X with (**projection**) maps [3] $p_\alpha : X \longrightarrow X_\alpha$, such that every family $f_\alpha : Z \longrightarrow X_\alpha$ of maps from some other topological space **factors through** the p_α *uniquely*, in the sense that there is a unique $f : Z \longrightarrow X$ such that

$$f_\alpha = p_\alpha \circ f \qquad \text{(for all } \alpha\text{)}$$

Pictorially, one says that **all triangles commute** in the diagram [4]



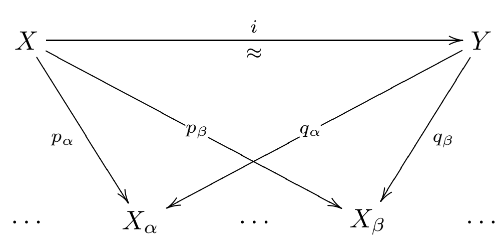
Remark: Why not some *other* diagrams, instead? This is a fair question. Indeed, reversing *all* the arrows turns out to be reasonable, defining a **coproduct**. Other choices of arrow directions turn out to be silly. We examine these possibilities after talking about products.

Claim: There is at most one product of given spaces X_α, up to *unique* isomorphism.

That is, given two products, X with projections p_α and Y with projections q_α, there is a unique isomorphism $i : X \longrightarrow Y$ respecting the projections in the sense that

$$p_\alpha = q_\alpha \circ i \qquad \text{(for all } \alpha\text{)}$$

That is, there is a unique isomorphism $i : X \longrightarrow Y$ such that all triangles commute in the diagram

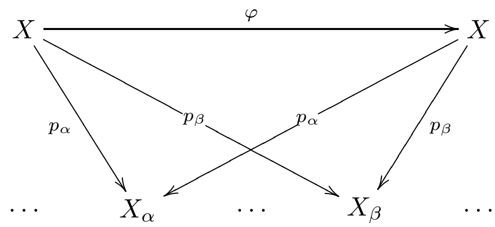


Proof: First, we prove that products *have no (proper) endomorphisms*, meaning that the *identity map* is

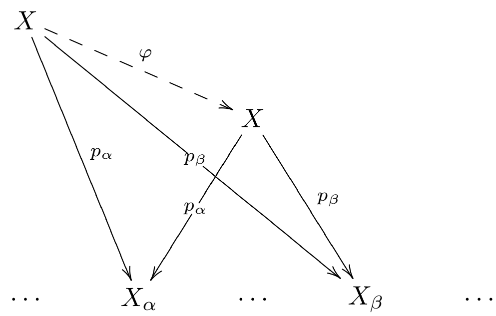
[3] Throughout this discussion *map* means *continuous map*, although much of what is said will not use continuity.

[4] Often an arrow is drawn with a dotted line rather than a solid line to suggest that it is a consequence of or is derived from the other maps in the diagram.

the only map $\varphi : X \longrightarrow X$ making all triangles commute in the diagram

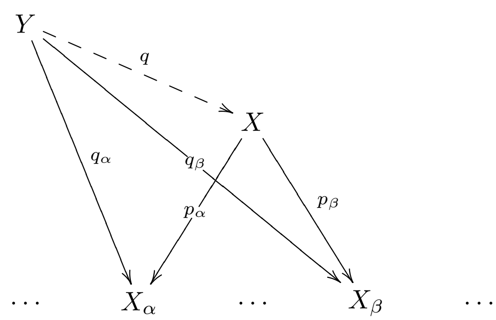


Indeed, using X and the p_α in the role of Z and f_α in the defining property of the product, we have a unique φ making all triangles commute in



The identity map certainly meets the requirement on φ, so, by uniqueness, the identity map on X is the *only* such map φ.

Next, let Y be another product, with projections q_α. Letting Y take the role of Z and q_α the role of f_α in the defining property of the product X (with its p_α's), we have a *unique* $q : Y \longrightarrow X$ such that all triangles commute in



Reversing the roles of X and Y (and their projections), we also have a *unique* $p : X \longrightarrow Y$ such that (this time in symbols rather than the picture) $p_\alpha = q_\alpha \circ p$ for all α.

Then $p \circ q : Y \longrightarrow Y$ is an endomorphism of Y respecting the projections q_α, since

$$q_\alpha \circ (p \circ q) = (q_\alpha \circ p) \circ q = p_\alpha \circ q = q_\alpha$$

Thus, $p \circ q$ must be the identity on Y. Similarly, $q \circ p$ is the identity on X. Thus, p and q are mutual inverses, so both are isomorphisms. And we did have *uniqueness*, along the way. ///

Remark: We did not use any features of topological spaces nor of continuous maps in this proof. Instead, the quantification over all Z and maps f_α indirectly described what was happening. Thus, we discover that a similar result holds for any prescribed collection of things with prescribed maps between them that allow *composition*, etc. In particular, for our present purposes we have shown that the product is unique up to *continuous* isomorphism, not just *set* isomorphism.

Remark: Since there is at most one product, the issue of figuring out what it *is* will be simpler than if there were many possibilities. Further, we can use the mapping properties to find hints about a *construction*, thus proving existence.

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2. *Construction of products of sets*

Before addressing the *topology* on the product, we first construct it as a *set*. Of course, we secretly know that it is the usual Cartesian product, but it would be interesting to see whether we can see this directly from the mapping properties, rather than pointlessly and boringly merely verify that the Cartesian product fits (which we do at the end).

Note that the uniqueness proof just given applies immediately to *sets without topologies*, as well. That is, the same diagrams but with objects just sets and maps arbitrary set maps, *define* a product X (with projections p_α) of non-empty *sets* X_α, and the same argument proves that there is at most one such thing (up to unique isomorphism). And, yes, we secretly know that the product is the usual Cartesian, and the projections are the obvious things, but we also want to see how one might be *led to* this understanding.

To investigate properties of a product X of non-empty sets X_α (with projections p_α), we should consider various sets Z and maps $f_\alpha : Z \longrightarrow X_\alpha$ to see what we learn about X as by-products. In this austere setting, there are not very many choices that suggest themselves, but, on the other hand, there are not many *wrong* choices available, either.

For example, given $x \in X$, we have the collection of all projections' values $p_\alpha(x)$, with unclear relations (or none at all) among these values. For example, to see whether things get mashed down, and as an exercise in technique, we could try to prove

Claim: For $x \neq y$ both in X, there is at least one $\alpha \in A$ such that $p_\alpha(x) \neq p_\alpha(y)$.

Proof: Suppose that $p_\alpha(x) = p_\alpha(y)$ for all $\alpha \in A$. Let $S = \{s\}$ be a set with one element, and define $f_\alpha : S \longrightarrow X_\alpha$ by

$$f_\alpha(s) = p_\alpha(x) \qquad (= p_\alpha(y), \text{ also})$$

Then, by definition, there is a unique $f : S \longrightarrow X$ such that $f_\alpha = p_\alpha \circ f$ for all α. But notice that f defined by $f(s) = x$ and also f defined by $f(s) = y$ have this property. By uniqueness of f, we have $x = f(s) = y$. ///

From the other side, we can wonder whether *all* possible collections of values of projections can occur. Intuitively (and secretly knowing the answer in advance) we might doubt that there are any constraints, but we are obliged to demonstrate this by exhibiting maps.

Claim: Given choices $x_\alpha \in X_\alpha$ for all $\alpha \in A$, there is x in the product such that $p_\alpha(x) = x_\alpha$ for all α. (And by the previous claim this x is *unique*.)

Proof: Again, let $S = \{s\}$ be a set with a single element, and define $f_\alpha : S \longrightarrow X_\alpha$ by $f_\alpha(s) = x_\alpha$. Then there exists a unique $f : S \longrightarrow X$ such that $f_\alpha = p_\alpha \circ f$. That is

$$x_\alpha = f_\alpha(s) = (p_\alpha \circ f)(s) = p_\alpha(f(s))$$

The element $x = f(s)$ is the desired one. ///

In fact, these two claims, *with their proofs*, suggest that the product X *is exactly* the collection of all choices of families $\{f_\alpha : \alpha \in A\}$ of functions $f_\alpha : \{s\} \longrightarrow X_\alpha$, and α^{th} projection given by

$$p_\alpha \;:\; \{f_\alpha : \alpha \in A\} \longrightarrow f_\alpha(s)$$

This is correct, and by uniqueness we know that any other construction must give an isomorphic thing, but this can be simplified, since maps from a one-element set are entirely determined by their images. Thus, finally, we have been led back to the Cartesian product

$$X = \{\{x_\beta : \beta \in A\} \ : \ x_\beta \in X_\beta\}$$

and usual projections

$$p_\alpha(\{x_\beta : \beta \in A\}) = x_\alpha$$

And, finally, we give the trivial proof of

Claim: The Cartesian product and usual projections are a product for sets.

Proof: Given a family of maps $f_\alpha : Z \longrightarrow X_\alpha$, define $f : Z \longrightarrow X$ by

$$f(z) = \{f_\beta(z) : \beta \in A\}) \in X$$

This meets the defining condition, and is visibly the only map that will do so. ///

Remark: The point here was that there is no alternative, up to (unique!) isomorphism, and that some reasonable considerations based on the mapping-property definition can lead us to a construction.

Remark: The little fact that the collection of set maps φ from $S = \{s\}$ to a given set Y is isomorphic to Y (via $\varphi \longrightarrow \varphi(s)$) is noteworthy in itself.

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3. *Construction of product topologies*

So, at last, what topology does the mapping-property definition require on the Cartesian product X of the underlying *sets* for non-empty topological spaces X_α?

One should be aware that it is possible to apparently define non-existent (impossible) objects by mapping properties, and the impossibility may not be immediately clear. Thus, one should have respect for the problem of *constructing* objects whose uniqueness (if they exist) is much easier.

First, all the projections $p_\alpha : X \longrightarrow X_\alpha$ defined as expected by

$$p_\alpha(\{x_\beta : \beta \in A\}) = x_\alpha$$

must be continuous. That is,[5] for every open set U_α in X_α, $p_\alpha^{-1}(U_\alpha)$ is open in X. Note that

$$p_\alpha^{-1}(U_\alpha) = \prod_{\beta \in A} U_\beta \qquad (\text{where } U_\beta = X_\beta \text{ for } \beta \neq \alpha)$$

Thus, the topology on X must contain *at least* these sets as opens, which does entail that the topology include finite intersections of them, which are exactly[6] sets of the form

$$\prod_{\beta \in A} U_\beta \qquad (\text{with } U_\beta \text{ open in } X_\beta, \text{ and } U_\beta = X_\beta \text{ for all but finitely-many } \beta)$$

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[5] Recall the definition of continuity of a map $\varphi : A \longrightarrow B$ of topological spaces, namely, that the inverse image $\varphi^{-1}(U)$ of every open U in B is open in A.

[6] This fact may take a moment's reflection to verify.

And arbitrary unions of these finite intersections must be included in the topology, and so on. [7] [8] Thus, the product topology is *at least as fine as* the topology generated by the sets $p_\alpha^{-1}(U_\alpha)$. [9]

From the other side, the condition concerning maps from another space Z gives a constraint on how *coarse* the topology must be, as follows. Given a family of continuous maps $f_\alpha : Z \longrightarrow X_\alpha$, the corresponding $f : Z \longrightarrow X$ must be continuous, which requires by definition that $f^{-1}(U)$ must be open in Z for all opens U in X. If there were *too many* opens U in X this condition could not be met. So the topology on X must be *at least as coarse* as would be allowed by $f^{-1}(U)$ being open in Z for all $f : Z \longrightarrow X$. Yes, this is a less tangible constraint, since it is hard to visualize the quantification over all $f : Z \longrightarrow X$.

It is reasonable to *hope* that the explicit topology on X generated by the inverse images $p_\alpha^{-1}(U)$ can be *proven* to be sufficiently coarse to meet the second condition.

To prove continuity of $f : Z \longrightarrow X$, thus proving that the topology on X is sufficiently coarse, it suffices to prove that $f^{-1}(U)$ is open for all opens U *in a sub-basis.* [10] And, happily, by $p_\alpha \circ f = f_\alpha$,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = f_\alpha^{-1}(U_\alpha)$$

which is open by the continuity of f_α. So f is continuous, which is to say that the topology on X is coarse enough (not too fine), so succeeds in being suitable for a product. ///

Remark: Thus, the product topology must be *fine enough* so that inverse images $p_\alpha(U_\alpha)$ in X of projections $p_\alpha : X \longrightarrow X_\alpha$ are open, and *coarse enough* so that inverse images $f^{-1}(U)$ in Z of induced maps $f : Z \longrightarrow X$ are open.

Remark: The *main* issue here is *existence* of any topology at all that will work as a product. By contrast, the *uniqueness* of products (if they exist) was proven earlier, in a standard (even clichéd) mapping-property fashion. That is, the uniqueness *up to unique isomorphism* asserts that if X and Y were two products, they must be (uniquely) homeomorphic.

Remark: To repeat: the question of *existence* of a product topology can be viewed as being the question of whether or not the topology generated by the sets $p_\alpha^{-1}(U_\alpha)$ might accidentally be *too fine* for all induced maps $Z \longrightarrow X$ to be continuous. That is, the continuity of the projections and the continuity of the induced maps $Z \longrightarrow X$ are *opposing* constraints. The for-general-reasons uniqueness tells us *a priori* that there is *at most one* simultaneous solution, so the question is whether or not there is *any.* In this case, it turns out that these conflicting constraints *do* allow a common solution. Still, it does happen in other circumstances that two opposing conditions allow *no* simultaneous satisfaction.

4. *Why not something else, instead?*

[7] In fact, further finite intersections or arbitrary unions produce no new sets, so all opens are already expressible as unions of finite intersections of the sets of the form $p_\alpha^{-1}(U_\alpha)$.

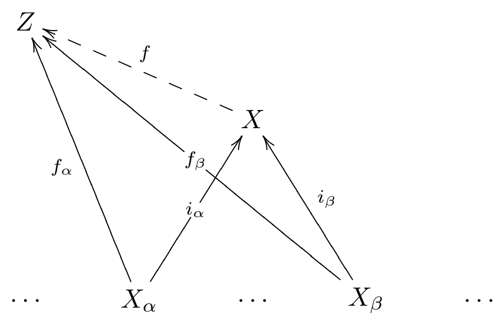
[8] We could say that the product topology (assuming that it exists) is *generated by* these sets $p^{-1}(U_\alpha)$, but it is more usual (and less explanatory) to say that these sets are a *sub-basis* for the topology. Recall that a *basis* for a topology is a set of opens such that *any* open set is a union of some of the given opens. A *sub-basis* for a topology is a set of opens so that any open is a union of *finite intersections* of the given opens.

[9] Recall that one topology on a set X is *finer* than a second topology on the same set if it contains all the opens of the second topology, and possibly more. And then the second topology is said to be *coarser* than the first.

[10] Again, a sub-basis for a topology is any family of opens such that *every* open is a union of finite intersections from the family. The reason this is sufficient is that the inverse image map $U \longrightarrow f^{-1}(U)$ of sets preserves intersections and unions, unlike functions themselves.

For context, we should ask ourselves (at least) what happens if some or all of the arrows in the diagram defining the product are *reversed*. On the face of it, this might be an idle question, but still deserves an answer. The fact that we succeeded in characterizing products by mapping properties was *not* the final goal, after all. Rather, this was to have been an exercise illustrating a larger point.

As mentioned earlier, when *all* the arrows are reversed, the object defined is a **coproduct**. That is, given a family $\{X_\alpha : \alpha \in A\}$, a *coproduct* of the X_α is an X with maps[11] $i_\alpha : X_\alpha \longrightarrow X$ such that, for all Z and maps $f_\alpha : X_\alpha \longrightarrow Z$, there is a unique $f : X \longrightarrow Z$ such that every f_α factors through f, that is, such that $f_\alpha = f \circ i_\alpha$ for all α. In diagrams this asserts that there exists a unique $f : X \longrightarrow Z$ such that all triangles commute in



Remark: We are scrupulously avoiding saying what kind of objects X_α are involved, and what kind of maps are involved. The objects might be sets and the maps set maps, or the objects topological spaces and the maps continuous, or the objects abelian groups and the maps group homomorphisms, and so on.

Claim: For sets and set maps, coproducts are *disjoint unions*. That is, given sets X_α for $\alpha \in A$, the coproduct of the X_α is the *disjoint* union[12] $X = \bigsqcup_{\alpha \in A} X_\alpha$ with literal inclusions $i_\alpha : X_\alpha \longrightarrow X$.

We leave the discussion or proof of this to the reader. In any case, disjoint unions are of some interest, though perhaps not as much as products. For contrast, we also consider

Claim: For abelian groups and group homomorphisms, coproducts are *direct sums*. That is, given abelian groups X_α for $\alpha \in A$, the coproduct of the X_α is the direct sum

$$X = \bigoplus_{\alpha \in A} X_\alpha = \{\{x_\beta : \beta \in A\} \;:\; \text{all but finitely-many } x_\beta\text{'s are } 0 \in X_\alpha\}$$

with inclusions $i_\alpha : X_\alpha \longrightarrow X$ by

$$i_\alpha(x_\alpha) = \{y_\beta : y_\beta = 0 \in X_\beta \text{ for } \beta \neq \alpha, \;\; \text{and } y_\alpha = x_\alpha\}$$

We also leave the proof or discussion of this second claim to the interested reader.

Remark: The difference between these two coproducts is in contrast to the analogues for *products*. That is, the underlying *set* of a product of groups *is* the (set) product of the underlying sets. By contrast, the underlying set of a coproduct of abelian groups is *not* the (set) coproduct of the underlying sets.

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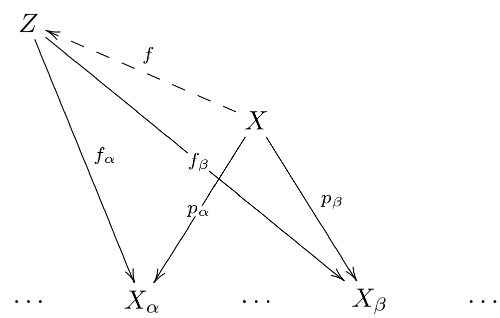
[11] These maps i_α would usually be called *inclusions* or *imbeddings*, but use of these terms might prejudice the reader. They will indeed *turn out* to be injections.

[12] The notion of *disjoint union* is that, if by chance the sets X_α as they naturally occur have non-trivial intersections, we add artifical labels or play other tricks to make *isomorphic* sets which are pairwise disjoint, and *then* take the union.

Now we consider further variations on the arrows. *None of these turns out to be interesting*, as it happens.

For example, if we take the defining diagram for a product of X_α's, but only reverse the dotted arrow, we require that there is X and maps $p_\alpha : X \longrightarrow X_\alpha$ such that for all families $f_\alpha : Z \longrightarrow X_\alpha$ there is a unique $f : X \longrightarrow Z$ (this is the arrow reversal) such that $p_\alpha = f_\alpha \circ f$. (This family of conditions is the only possible thing to require of a similar sort, given the directions of the arrows.) That is, we require the commutativity of all triangles in diagrams



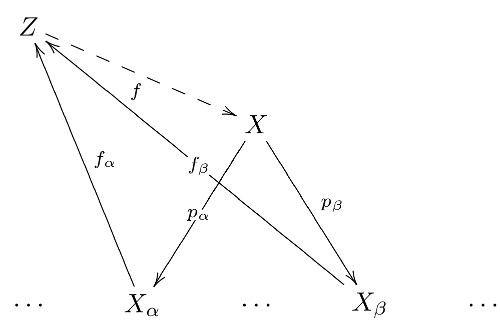
Claim: For non-empty sets X_α, if any one has at least two elements, then there is no such set X (with set maps p_α).

Proof: Much as in our discussion of products of sets, we may take $Z = \{z\}$, a set with a single element z, and for a selection of elements $x_\alpha \in X_\alpha$ for all $\alpha \in A$, define $f_\alpha : Z \longrightarrow X_\alpha$ by $f_\alpha(z) = x_\alpha$. Supposing that X is a non-empty set, this implies that

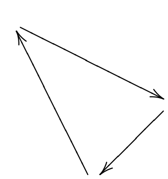
$$p_\alpha(X) = (f_\alpha \circ f)(X) \subset f_\alpha(Z) = \{x_\alpha\}$$

for all α. But when X_α has two or more elements, this is impossible, since p_α is surjective. ///

As another type of uninteresting outcome, take the defining diagram for a product of X_α's, and reverse the f_α's, that is, we consider diagrams



Now there is a different failure, namely that all the patterns of arrows in triangles are of the *cyclic* form



which suggests no obvious commutativity condition. We leave other choices of conditions to the bored or whimsical reader.

In summary, we have a mapping condition characterization of *products*, and also, incidentally, of *coproducts*. This explains how these objects *interact* with other objects, which is of far greater import than their internal construction.