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Bigger diagrams for solenoids, more automorphisms, colimits

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Bigger diagrams make visible more automorphisms: more precisely, bigger diagrams *with the same limit* make visible more automorphisms *of the same object*. In the case of the 2-solenoid, this means that we will find a copy of the 2-**adic rational numbers**[1] $\mathbb{Q}_2$ acting on the 2-solenoid, rather than merely the 2-**adic integers** $\mathbb{Z}_2$. This is a big change in the sense that $\mathbb{Q}_2$ is *non-compact*, while $\mathbb{Z}_2$ is *compact*. We had already seen that as a $\mathbb{R}\times\mathbb{Z}_2$ -space

$$\text{2-solenoid} \approx (\mathbb{R}\times\mathbb{Z}_2)/\mathbb{Z}^\Delta$$

where $\mathbb{Z}^\Delta$ is the diagonally imbedded copy of $\mathbb{Z}$. Having found a larger group of automorphisms, we will find that

$$\text{2-solenoid} \approx (\mathbb{R}\times\mathbb{Q}_2)/\mathbb{Z}[1/2]^\Delta$$

as $\mathbb{R}\times\mathbb{Q}_2$ -space, *and* that the diagonal copy $\mathbb{Z}[1/2]^\Delta$ of

$$\mathbb{Z}[1/2] = \mathbb{Z} + \frac{1}{2}\cdot\mathbb{Z} + \frac{1}{4}\cdot\mathbb{Z} + \frac{1}{8}\cdot\mathbb{Z} + \ldots$$

(the rational numbers with denominators restricted to be powers of 2) is still *discrete*[2] in the product $\mathbb{R}\times\mathbb{Q}_2$, and the 2-adic rationals $\mathbb{Q}_2$ are presented naturally as a *(strict) colimit* of topological groups

$$\mathbb{Z}_2 \subset \frac{1}{2}\cdot\mathbb{Z}_2 \subset \frac{1}{4}\cdot\mathbb{Z}_2 \subset \frac{1}{8}\cdot\mathbb{Z}_2 \subset \ldots$$

That is, at the level of *sets*, we have an *ascending union*. To be sure that we give this ascending union a suitable topology, consideration of mapping properties is wise.

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1. *Bigger diagrams, more automorphisms*

Incidental to refining our viewpoint on the 2-solenoid, we should verify that many different (related, of course) diagrams can easily give the same limit object. The slogan here is that *cofinal limits are (naturally) isomorphic*. We only prove the simple special case of this we need for immediate use, below. A fuller version of this issue will arise with *wider solenoids*, approaching the *adeles*, shortly.

In particular, there is the theme of finding larger diagrams that have no *bottom* object (but give the same limit), with motivation of finding larger automorphism groups. Discrete diagrams *with* bottom objects often give *compact* limits, and this may mask interesting non-compact automorphism groups whose quotients are (nevertheless) compact.

So far, we have the 2-solenoid X as a projective limit fitting into a diagram

[1] We will review the classical definition of the p-adic rationals and integers shortly. For the moment, we simply use these names for the things that appear, without pretending to have proven that the naming is apt.

[2] As usual, a subset Y of a topological space X is *discrete* if each point y of Y has a neighborhood U in X such that $U\cap Y = \{y\}$.

$$X \qquad \cdots \longrightarrow \mathbb{R}/4\mathbb{Z} \longrightarrow \mathbb{R}/2\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z}$$

Given a point x on the solenoid, let x_n be its projection to $\mathbb{R}/2^n\mathbb{Z}$, and we think of such a point as being a compatible family of points on the respective circles, written

$$x \ldots \to x_2 \to x_1 \to x_0$$

Let's review the way we found

$$\mathbb{Z}_2 = \lim\,(\ldots \to \mathbb{Z}/8 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to \mathbb{Z}/1)$$

as a group acting on the 2-solenoid. As earlier, given a point x on X, we act by an element $r \in \mathbb{R}$ on all circles simultaneously, to make the new 0^{th} projection $0 \in \mathbb{R}/\mathbb{Z}$. That is, the *new* values

$$x \ldots \to x_2 \to x_1 \to x_0 = 0$$

must actually be in $\mathbb{Z}$, and form a compatible family inside

$$\cdots \xrightarrow{\text{mod } 8} \mathbb{Z}/8 \xrightarrow{\text{mod } 4} \mathbb{Z}/4 \xrightarrow{\text{mod } 2} \mathbb{Z}/2 \xrightarrow{\text{mod } 1} \mathbb{Z}/1$$

But in the diagram defining the 2-solenoid there is no compulsion to stop at the circle $\mathbb{R}/\mathbb{Z}$. If we want, we could continue to the right with ever-shrinking circles, as in

$$\cdots \longrightarrow \mathbb{R}/4\mathbb{Z} \longrightarrow \mathbb{R}/2\mathbb{Z} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}/\tfrac{1}{2}\mathbb{Z} \longrightarrow \mathbb{R}/\tfrac{1}{4}\mathbb{Z} \longrightarrow \mathbb{R}/\tfrac{1}{8}\mathbb{Z} \longrightarrow \cdots$$

[1.0.1] Claim: The (projective) limit[3] of this diagram is naturally isomorphic to the limit of the original diagram.

[1.0.2] Remark: This is not at all surprising at a heuristic level, but it is an example of an important general fact, that *cofinal limits are isomorphic*. The general case is also important, but it is useful to give a quick proof in a more limited family of special cases, too.

Proof: Let X be a projective limit fitting into a commutative diagram

$$X \qquad \cdots \longrightarrow X_1 \longrightarrow X_0$$

and consider also an enlarged diagram with projective limit Y:

$$Y \qquad \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow \cdots \longrightarrow X_{-n} \longrightarrow \cdots$$

We claim that there is a natural isomorphism $X \to Y$, induced from the commutative diagram

$$\begin{array}{ccccccccc} X & \cdots \longrightarrow & X_1 & \longrightarrow & X_0 & & & & \\ & & \| & & \| & & & & \\ Y & \cdots \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow \cdots \longrightarrow & X_{-n} & \longrightarrow \cdots \end{array}$$

[3] One could also wonder what sort of limit this diagram has *to the right*, meaning an object with compatible maps *from* all the ever-shrinking circles. Suitable choices, illustrated just a little later, do lead to the useful notion of a *colimit*.

First, to make a map from Y to the projective limit X is exactly to have a compatible family of maps from Y to the X_n with $n \geq 0$. The projections of Y to the X_n with $n \geq 0$ already provide this, and we ignore the X_i with $i < 0$ at this moment. On the other hand, to get a map from X to Y is to give a compatible family of maps from X to all the X_n, now with $n \in \mathbb{Z}$. For $n \geq 0$ the projections of X to X_n work. For $-n < 0$, in fact, there are many possibilities. For example, map X to X_0 and then map to X_{-n} by the transition maps used in the diagram for Y.

Thus, we obtain *unique* maps $f : Y \to X$ and $g : X \to Y$ compatible with all the projections and equalities. Then $f \circ g : Y \to Y$ is a self-map of Y preserving all the projections, so, by the uniqueness of the projective limit, must be the identity map. Similarly, $g \circ f$ is the identity on X. Thus, $X \approx Y$. ///

[1.0.3] Remark: Again, the purely arrow-theoretic proof captures whatever information and conditions are implicit in the objects and maps we consider, such as topologies and continuity, group homomorphisms, and so on.

The larger diagram for the same object makes more automorphisms visible, as follows.

Given a point

$$x \ldots \to x_1 \to x_0 \to x_{-1} \to \ldots$$

in the larger diagram, since there is no obvious *bottom circle* to normalize, we have the further auxiliary choice of an integer n, and rotate $x_n \in \mathbb{R}/2^n\mathbb{Z}$ to 0. To help us remember what we're doing, let's take $\mathbb{Z} \ni -n \leq 0$, and let $\mathbb{R}$ act by $x_i \to x_i + r$ for all indices i, with r chosen to rotate x_{-n} to 0 in $\mathbb{R}/2^{-n}\mathbb{Z}$. Thus, we have

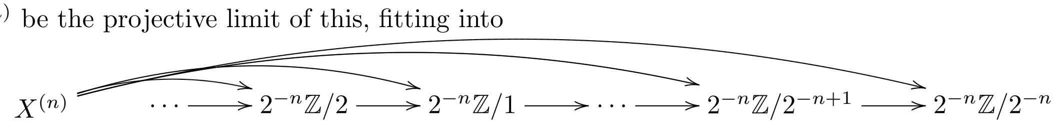
$$\ldots \to x_1 \to x_0 \to x_{-1} \to \ldots \to x_{-n} = 0 \to 0 \to 0 \to \ldots$$

since the arrows are group homomorphisms. That is, at and after the $-n^{th}$ place, all the (rotated) x_i are simply 0.

Thus, $x_{-n} = 0 \in 2^{-n}\mathbb{Z}/2^{-n}\mathbb{Z}$, and there are exactly 2 choices for x_{-n+1}, namely 0 and $2^{-n} \bmod 2^{-n+1}$. For each of these 2 choices, there are 2 choices of x_{-n+2}, and so on. Note that the choice $x_{-n} = 0$ on the n^{th} circle $\mathbb{R}/2^{-n}\mathbb{Z}$ means that x_{-n+i} is in $2^{-n}\mathbb{Z}$ modulo $2^{-n+i}\mathbb{Z}$. The collection of all such compatible families for a *fixed* choice of $-n$ fits together as

$$\ldots \to 2^{-n}\mathbb{Z}/4 \to 2^{-n}\mathbb{Z}/2 \to 2^{-n}\mathbb{Z}/1 \to 2^{-n}\mathbb{Z}/\frac{1}{2}\mathbb{Z} \to 2^{-n}\mathbb{Z}/\frac{1}{4}\mathbb{Z} \to \ldots \to 2^{-n}\mathbb{Z}/2^{-n}\mathbb{Z} \approx \{0\}$$

Let $X^{(n)}$ be the projective limit of this, fitting into



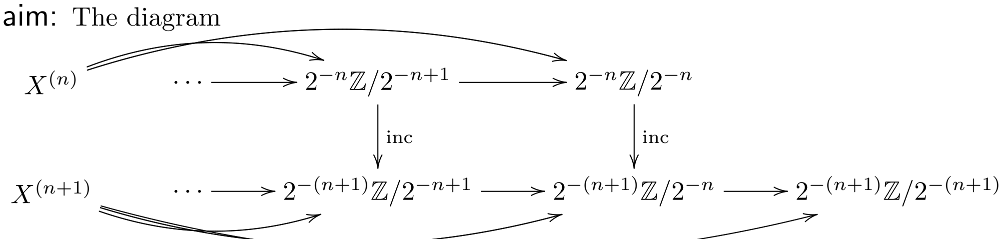
At least heuristically, we can give $X^{(n)}$ a more suggestive name and notation, specifically

$$2^{-n}\mathbb{Z}_2 = X^{(n)}$$

but we should not accidentally presume too much from the notation.

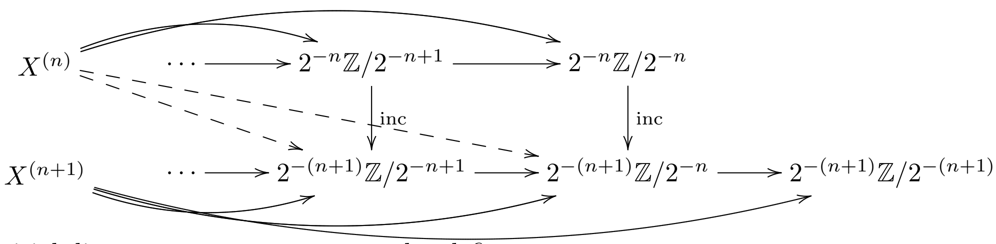
It is easy to *imagine* that the family of these diagrams fits together, giving an ascending chain of larger-and-larger limits. Indeed,

[1.0.4] Claim: The diagram



induces a unique *injective* map [4] $X^{(n)} \to X^{(n+1)}$ compatible with all the projections (where the vertical maps are the obvious inclusions).

Proof: Again, to give a map to a projective limit is to give a compatible family of maps to the things in the limit. Thus, by composition with the inclusions, we obtain the dashed arrows



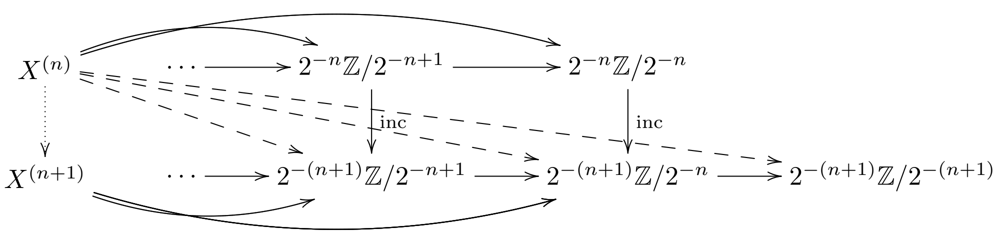
Since the initial diagram commutes, we can also define a map

$$2^{-n}\mathbb{Z}_2 = X^{(n)} \to 2^{-(n+1)}\mathbb{Z}/2^{-(n+1)}$$

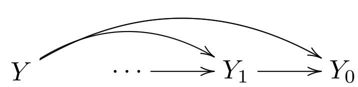
by composition with *any choice of* inclusion map from the top row to the bottom. Thus, we have a unique induced dotted arrow

$$2^{-n}\mathbb{Z}_2 = X^{(n)} \to X^{(n+1)} = 2^{-(n+1)}\mathbb{Z}_2$$

in the commuting diagram



We must prove that the induced map is injective. [5] First, we claim that an element y in a projective limit



is 0 if and only if all its projections y_i are 0. This reviews a minor mapping-property trick applicable to objects that have an underlying structure of *set*. That is, the elements of a set W are in bijection with the maps of a singleton set $S = \{s\}$ to W, simply by taking a map f to $f(s)$. Thus, since limits of topological groups have the same underlying set as the corresponding limit of *sets*, elements of the limit Y are given by compatible families of maps $S \to Y_i$. If all these are 0, then $f(s) = 0$ is certainly *a* compatible map to the limit. By *uniqueness*, there is no *other* image possible. The converse is immediate.

Thus, given non-zero $x \in X^{(n)}$, at least one projected image $x_i \in 2^{-n}\mathbb{Z}/2^{-n+i}$ is non-zero. The inclusion to $2^{-(n+1)}\mathbb{Z}/2^{-n+i}$ is still non-zero, so the image under the induced map to $X^{(n+1)}$ cannot be 0. Thus, the (abelian) group homomorphism $X^{(n)} \to X^{(n+1)}$ has trivial kernel, so is injective. ///

[4] As usual in our discussions, a *map* is implicitly continuous, and here is a group homomorphism. The arrow-theoretic nature of the argument carries these details along implicitly, and by its nature is applicable to many other situations as well.

[5] Depending on one's outlook, this might be a moment to introduce the purely mapping-theoretic version of injective maps, namely *monomorphisms*. We won't take this approach, but will give the definition: a map $i : X \to Y$ is a *monomorphism* (in whatever category) if for all maps $f, g : Z \to X$, the composites $i \circ f$ and $i \circ g$ are equal only if $f = g$.

Thus, we have a family of inclusions

$$\begin{array}{ccccccc} X^{(0)} & \xrightarrow{\text{inc}} & X^{(1)} & \xrightarrow{\text{inc}} & X^{(2)} & \xrightarrow{\text{inc}} & \ldots \\ \| & & \| & & \| & & \\ \mathbb{Z}_2 & & \frac{1}{2}\cdot\mathbb{Z}_2 & & \frac{1}{4}\cdot\mathbb{Z}_2 & & \end{array}$$

of groups acting on the 2-solenoid. Of course the action of $X^{(n+1)}$ matches that of $X^{(n)}$ when restricted to $X^{(n)}$, so we *apparently* have an action on the 2-solenoid of the **ascending union**

$$\mathbb{Q}_2 = \bigcup_{n=0}^{\infty} 2^{-n}\mathbb{Z}_2 = \bigcup_{n=0}^{\infty} X^{(n)}$$

[1.0.5] Remark: Several expected things really are true: the ascending union $\mathbb{Q}_2$ has a reasonable topology, and acts continuously on the 2-solenoid. However, these conclusions do not follow as easily, or superficially, from general mapping properties. That is, colimits do not behave as well (for our purposes) as do limits. We will look at these issues just below. For the moment, we continue without worrying too much.

Next, we determine the *isotropy subgroup* of the point 0 in the 2-solenoid, under the action of $\mathbb{R}\times\mathbb{Q}_2$. Recall that $r\in\mathbb{R}$ acts by

$$r\cdot(\ldots\to x_i \bmod 2^i\mathbb{Z}\to\ldots) = \ldots\to r+x_i \bmod 2^i\mathbb{Z}\to\ldots$$

Similarly, each $y\in\mathbb{Q}_2$ is of the form (for some n, depending on y)

$$\ldots\to y_i \bmod 2^i\mathbb{Z}\to\ldots\to y_{-n+1} \bmod 2^{-n+1}\mathbb{Z}\to y_{-n}=0 \bmod 2^{-n}\mathbb{Z}\to 0\to 0\to\ldots$$

with y_{-n+i} lying in $2^{-n}\mathbb{Z}/2^{-n+i}\mathbb{Z}$. With this way of presenting it, the action on x in the 2-solenoid is straightforward, namely

$$y\cdot x = \big(\ldots\to y_i+x_i \bmod 2^i\mathbb{Z}\to\ldots\big)$$

Already $\mathbb{R}\times\mathbb{Z}_2$ was transitive, so certainly $\mathbb{R}\times\mathbb{Q}_2$ is transitive. The isotropy group of the point $x=0$ in the 2-solenoid is the collection of $r\in\mathbb{R}$ and $y\in\mathbb{Q}_2$ such that

$$r+y_i\in 2^i\mathbb{Z} \quad (\text{for all } i\in\mathbb{Z})$$

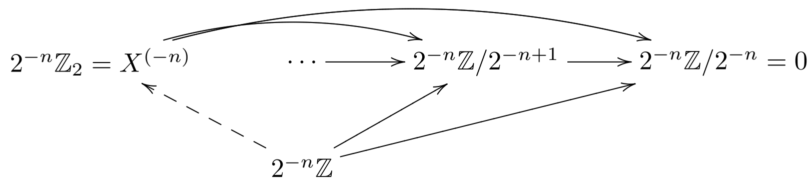
where for each $y\in\mathbb{Q}_2$ there is an integer $n\ge 0$ such that all y_i lie in $2^{-n}\mathbb{Z}$. For fixed y with associated n, taking $i=-n$, since $y_{-n}\in 2^{-n}\mathbb{Z}$, we find

$$r\in -y_{-n}+2^{-n}\mathbb{Z} = 2^{-n}\mathbb{Z}$$

Then for all indices $0\ge i\in\mathbb{Z}$, by the isotropy condition,

$$y_{-n+i} = -r \quad (\text{in } 2^{-n}\mathbb{Z}/2^{-n+i}\mathbb{Z})$$

Thus, for all $n\ge 0$, we have the diagonal copy of $2^{-n}\mathbb{Z}$ imbedded in $X^{(n)}=2^{-n}\mathbb{Z}_2$ induced from the diagram



That is, for all $n\ge 0$ we have

$$\big(2^{-n}\mathbb{Z}\big)^{\Delta} = \{(\delta,-\delta):\delta\in 2^{-n}\mathbb{Z}\}\subset\mathbb{R}\times\mathbb{Q}_2$$

inside the isotropy group. Taking the ascending union, we find the diagonal copy of $\mathbb{Z}[1/2]$ as exactly the isotropy group. Thus, as $\mathbb{R} \times \mathbb{Q}_2$ -spaces,

$$\text{2-solenoid} \approx (\mathbb{R} \times \mathbb{Q}_2)/\mathbb{Z}[1/2]^\Delta$$

[1.0.6] Remark: We have not yet shown that the diagonal copy of $\mathbb{Z}[1/2]$ is *discrete* in the product $\mathbb{R} \times \mathbb{Q}_2$. To do so, we need to see what topology $\mathbb{Q}_2$ has. [6] Certainly $\mathbb{Z}[1/2]$ is *not* discrete in $\mathbb{R}$, in fact is *dense*, even though $\mathbb{Z}$ without allowing 2's in the denominator *was* discrete in $\mathbb{R}$. It *will* also be the case that $\mathbb{Z}[1/2]$ is dense in $\mathbb{Q}_2$, and that it is only in the product $\mathbb{R} \times \mathbb{Q}_2$ that $\mathbb{Z}[1/2]$ becomes discrete.

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2. *Coproducts, colimits*

When we look at *colimits* and *coproducts* here, it is important to see that, while at an abstract level these things are just the arrows-reversed versions of limits and products, for many classes of naturally-occurring objects there is a sharp asymmetry. For example, while limits are *subobjects* of products, colimits are *quotients* of coproducts. In many situations, quotients are more abstract entities than are subobjects. This can be explained from a set-theoretic viewpoint, since elements of a subset are the same sort of thing as elements of the original set, since they *are* elements of the original set, while elements of *quotients* are *sets* of elements of the original.

In particular, in many cases colimits are *fragile*, and need further details or hypotheses to give us helpful outcomes. For example, while all subspaces of Hausdorff topological spaces are Hausdorff, *quotients* of Hausdorff topological spaces need not be Hausdorff.

In the simple case of circles and solenoids we're considering first, some of these themes are obscured by the very simplicity of the situation. Indeed, there are not many different compact, connected, one-dimensional manifolds: just circles. And these circles are themselves *groups*, and are *abelian*. But this careful preparation is intended to make our subsequent treatment of *surfaces* and other higher-dimensional examples less disconcerting.

The immediate goal is to give as graceful as possible a treatment of the topology on the *ascending union*

$$\mathbb{Q}_2 = \bigcup_{n=0}^{\infty} \frac{1}{2^n} \cdot \mathbb{Z}_2 = \bigcup_{n=0}^{\infty} \frac{1}{2^n} \cdot \Big(\lim_n \mathbb{Z}/2^n\mathbb{Z}\Big)$$

and to define a natural continuous action of $\mathbb{Q}_2$ on the 2-solenoid from those of the *limitands* [7] $2^{-n} \cdot \mathbb{Z}_2$ already treated. [8] Before doing this, however, we must make the effort to treat the problem as glibly as limits and products allowed, and we will find difficulties in the colimit situation unlike those for limits.

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[6] Yes, it is the metric topology that can be defined in the customary *ad hoc* fashion, but if we take *that* definition the problem becomes verifying that that is the same thing that we obtain here as the ascending union.

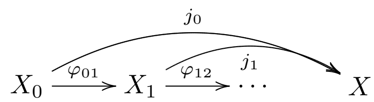
[7] *Limitand* is a made-up word, but serves its purpose. At least it has the pseudo-dignity of a pseudo-Latinate pseudo-etymology.

[8] It is possible to *misunderstand* the nature of $\mathbb{Q}_2$ when presented as an ascending union of $2^{-n}\mathbb{Z}_2$'s. The worst misunderstanding can be illustrated by a bad analogy, as follows. Returning to a more familiar setting, we can certainly write the real line $\mathbb{R}$ as an ascending union $\mathbb{R} = \bigcup_{n\geq 1} 2^n \cdot [-1, +1]$. Each interval $[-1, +1]$ is compact, and the dilations by powers of 2 are all homeomorphic to each other. Since $\mathbb{R}$ is certainly *not* compact, it would be very naive to think that expressing $\mathbb{R}$ in this fashion meant that $\mathbb{R}$ were somehow basically a compact interval. This is a *bad* analogy because closed intervals do not arise in the manner that the sets $2^{-n}\mathbb{Z}_2$ do, in many regards.

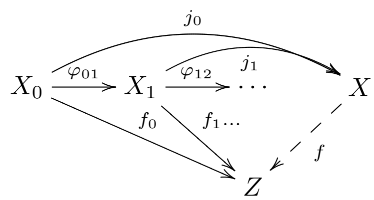
We need a *dual* notion to that of (projective) limit, namely **colimit**. [9] The definition can be obtained from the definition of limit [10] by reversing all the arrows.

[2.0.1] Remark: At a formal or abstract level, reversing the directions of arrows really does nothing. However, with the actual objects that occur in practice and are of interest to us, this reversal often matters a great deal. Again, for example, properties of *quotient* objects are often less predictable than properties of *sub*-objects. [11] Indeed, the smooth general use of products and limits is *not* matched by any similar smoothness in treatment of the arrow-reversed coproducts and colimits, below.

Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a family of objects with maps $\varphi_{i,i+1} : X_i \to X_{i+1}$. A **colimit** X of the X_i (and, implicitly, maps $\varphi_{i,i+1}$) is an object of the same sort, with *inclusion maps* [12] $j_i : X_i \to X$ giving (first) commutativity of the diagram



Second, it is required of X and the inclusion maps that, for all families of *compatible maps* $f_i : X_i \to Z$ (meaning $f_i = f_{i+1} \circ \varphi_{i,i+1}$ for all indices i), there is a unique $f : X \to Z$ giving a commutative diagram



For X meeting these conditions, write

$$X = \text{colim}_n X_n \qquad \text{(suppressing reference to the maps)}$$

Thus, as topological group,

$$\mathbb{Q}_2 = \text{colim}_n\, 2^{-n}\mathbb{Z}_2$$

[2.0.2] Example: For objects X_n which are simply *sets*, and assuming that the transition maps are *inclusions*, the colimit certainly does capture the notion of *ascending union*, since to give a set map from an ascending union is to give a family of maps on each X_n, with compatibility with respect to the inclusions. [13]

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[9] Actually, the only thing we really need for the moment is a very special case, a *strict colimit*. Also, a colimit may be called an *inductive limit*, and also possibly a *direct limit*.

[10] Again, when we say *limit* we mean *projective limit*, which is sometimes called *inverse limit*.

[11] We noted earlier that subspaces of Hausdorff spaces are Hausdorff, while quotients need not be. In a different vein, submodules of finitely-generated free modules over principal ideal domains are still free, while quotients certainly need not be.

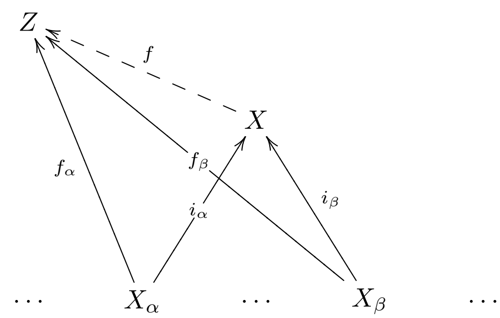
[12] These inclusion maps in colimits are opposite to the projection maps for limits. In many cases, such as when all $\varphi_{i,i+1}$ are injections, with perhaps further conditions, they are literally inclusions, but one should be cautious.

[13] Regarding ascending unions of *sets* with no further structure, with a family S_n with $S_n \subset S_{n+1}$, indexed by $n = 1, 2, 3, \ldots$, we probably have an intuitive belief that we can take the (ascending) union $S = \bigcup_n S_n$. However, from a careful foundational viewpoint, it is non-trivial to make sense of this, since unions must be taken *inside some larger set*.

As usual, if a colimit exists at all, then it is unique up to unique isomorphism. Thus, as usual, the more serious issue is *existence*, which needs a *construction*, either direct or indirect.

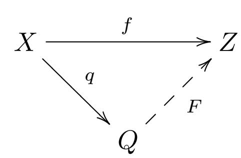
For present purposes, we will prove that colimits of *topological groups* can be constructed as corresponding colimits of *topological spaces*, with group structure hung on the set afterward. [14] First, we have a general result, applicable in any context where it makes sense, namely that often *colimits* can be constructed from *coproducts* as *quotients* by equivalence relations generated by the inclusion maps. We will use this preliminary result to prove that general colimits of topological spaces exist, from existence of coproducts. (Coproducts of topological spaces are disjoint unions with each piece given its own topology!)

Recall from earlier that, given a family of objects $\{X_\alpha : \alpha \in A\}$, a *coproduct* of the X_α is an X with maps $i_\alpha : X_\alpha \to X$ such that, for all Z and maps $f_\alpha : X_\alpha \to Z$, there is a unique $f : X \to Z$ such that every f_α factors through f, that is, such that $f_\alpha = f \circ i_\alpha$ for all α. In a diagram, this asserts that there exists a unique $f : X \to Z$ such that all triangles commute in



Also, we need a robust way to describe *quotients* of objects which do have an underlying set, without assuming too much further about what kind of things they are.

Let X be an object, and $\{x_\alpha : \alpha \in A\}$ and $\{y_\alpha : \alpha \in A\}$ two sets of elements of X. Then the **quotient** of X **by the relations** [15] $x_\alpha \sim y_\alpha$ (for all $\alpha \in A$) is another object Q with a map $q : X \to Q$ such that, all maps $f : X \to Z$ with $f(x_\alpha) = f(y_\alpha)$ for all $\alpha \in A$ *factor through* q uniquely. That is, there is $F : Q \to Z$ such that we have a commutative diagram



That is, Q is the *largest* quotient in which every x_α becomes equal to the corresponding y_α.

[2.0.3] Remark: As usual, if a quotient exists at all, it is unique up to unique isomorphism.

[2.0.4] Remark: With most familiar objects, quotients are readily *constructed*. For example, for a group G, the quotient by a family of relations $x_\alpha \sim y_\alpha$ (for x_α and y_α in G) is the usual group quotient of G by the intersection of all normal subgroups containing all the group elements $x_\alpha y_\alpha^{-1}$.

[2.0.5] Claim: For topological spaces, quotients exist.

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[14] While *group* colimits *can* be constructed from the *set* colimits of the underlying sets, the underlying set of a group *coproduct* is *not* the *set* coproduct. Specifically, set coproducts are *disjoint unions*, while *group* coproducts cannot possibly be disjoint unions.

[15] Here the symbol $\sim$ need *not* denote an equivalence relation!

Proof: This argument translates the present set-up into the usual *construction* of quotients of topological spaces, via equivalence relations. That is, given an equivalence relation R on a topological space X, the quotient X/R by R is the set X/R of equivalence classes of R, with natural quotient map $q : X \to X/R$, and with $U \subset X/R$ declared *open* if and only if $q^{-1}(U)$ is open in X. [16] Certainly this definition gives X/R a topology in which the quotient map is continuous. A given set of *required* relations $x_\alpha \sim y_\alpha$ for $\alpha \in A$, as in our general definition of *quotient*, may fail to be an equivalence relation, which requires the following adjustment. View $\sim$ in terms of its graph

$$\Gamma(\sim) = \{(x_\alpha, y_\alpha) \in X \times X : \alpha \in A\}$$

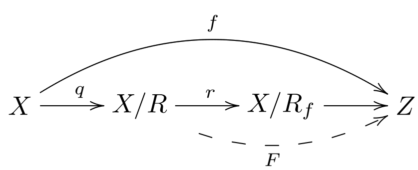
Define an associated *equivalence* relation R by taking the *graph* $\Gamma(R)$ of R to be the *intersection* of all graphs $\Gamma(S)$ of equivalence relations S containing $\Gamma(\sim)$, that is, [17]

$$\Gamma(R) = \bigcap_{S\,:\,\Gamma(S)\supset\Gamma(\sim)} \Gamma(S)$$

That is, in terms of graphs of equivalence relations, R is the *smallest* equivalence relation *containing* $\sim$. Let $f : X \to Z$ be a continuous map such that $f(x_\alpha) = f(y_\alpha)$ for all α. We must first show that f is actually constant on R-equivalence classes, so that at least as a *set* map f factors through the quotient $q : X \to X/R$. To this end, we cleverly observe that the relation R_f defined by

$$xR_fy \text{ if and only if } f(x) = f(y)$$

is an *equivalence* relation, and that its graph contains all pairs (x_α, y_α). So $\Gamma(R_f) \supset \Gamma(R)$, and we have a natural induced map $r : X/R \to X/R_f$, which is continuous since R_f is a stronger equivalence relation than R. The continuity of $f : X \to Z$ immediately tells us that the induced map $X/R_f \to Z$ is continuous, where X/R_f has the quotient topology in the usual sense. Thus, we get a (unique) $F : X/R \to Z$ giving a commutative diagram



This proves that the usual equivalence-relation definition of quotient topological space is a quotient object in our current sense. ///

It is not surprising that *some* general results work out for coproducts and colimits. For example, we can reduce existence of colimits to existence of coproducts and quotients.

[2.0.6] Claim: A colimit of a family

$$X_0 \xrightarrow{\varphi_{01}} X_1 \xrightarrow{\varphi_{01}} \cdots$$

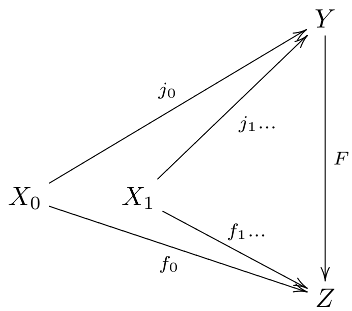
is a *quotient* of the *coproduct* $\coprod_n X_n$ (with accompanying inclusion maps $j_i : X_i \to \coprod_n X_n$) *by the relations*

$$j_m(x_m) \sim j_{m+1}(x_{m+1}) \quad \varphi_{m,m+1}(x_m) = x_{m+1}$$

[16] That this does give a topology on X/R follows easily from the definition of *topology*.

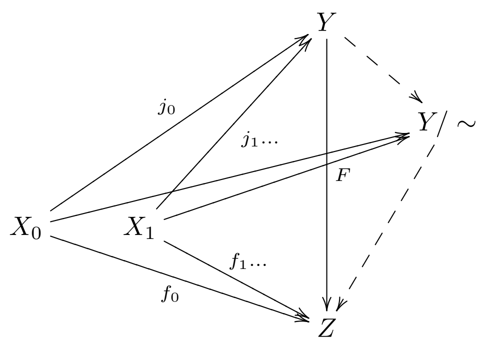
[17] Perhaps one should work the plausible exercise that an intersection of graphs of equivalence relations is again the graph of an equivalence relation.

Proof: This is the arrows-reversed version of the dual assertion, that *limits are subobjects of products.* Let Y be a *coproduct* of the X_n, with inclusions $i_n : X_n \to Y$. Given a compatible family $f_n : X_n \to Z$, let $F : Y \to Z$ be the (unique) map through which all the f_n factor. Diagrammatically, we have the commuting



Note that the map F exists regardless of compatibilities of the f_n with the transition maps $X_n \to X_{n+1}$. On the other hand, note that the inclusions i_n to the coproduct are *not* compatible with the maps $X_n \to X_{n+1}$.

For $x_m \in X_m$ and $x_{m+1} \in X_{m+1}$, and $\varphi_{m,m+1}(x_m) = x_{m+1}$, the compatibility of the f_m's with the transition maps is exactly that $f_{m+1}(\varphi_{m,m+1}(x_m)) = f_m(x_m)$. By the mapping-property definition of quotient, this implies that F factors uniquely through the quotient $Y/\sim$ of the coproduct Y by the given relations $\sim$.



This diagram shows that the quotient $Y/\sim$ is the colimit. ///

Thus, we have indirectly conjured up arbitrary colimits of topological spaces, from coproducts and quotients of them.

[2.0.7] Remark: To see that the argument above does not depend seriously upon the requirement that the index set be positive integers with the usual ordering, observe that we can define transition maps $\varphi_{mn} : X_m \to X_n$ for $m < n$ as the obvious composites

$$\varphi_{mn} = \varphi_{n-1,n} \circ \varphi_{n-2,n-1} \circ \ldots \circ \varphi_{m+1,m+2} \circ \varphi_{m,m+1}$$

Then the relation for the quotient is defined by

$$j_m(x_m) \sim j_n(x_n) \quad \text{for } m < n \text{ and } \varphi_{mn}(x_m) = x_n \text{ or } m > n \text{ and } \varphi_{nm}(x_n) = x_m$$

The the argument proceeds as before.

[2.0.8] Example: However, while every subspace of a Hausdorff topological space is again Hausdorff, *not* every quotient of a Hausdorff space is Hausdorff. For example, let X be the unit interval, and let Q be the quotient obtained by identifying [18] all points of the form $a/2^n$ for $a \in \mathbb{Z}$ and $0 \le n \in \mathbb{Z}$ in the interval. In this quotient, every neighborhood of every point contains all rationals $a/2^n$ from the interval, so is certainly not Hausdorff.

[2.0.9] Example: The abrupt identification of many points in the previous example can be accomplished gradually, perhaps subtly, in a *colimit*. For example, let X_0 be the unit interval $[0,1]$, and inductively define a family

$$X_0 \to X_1 \to X_2 \to \ldots$$

of successive quotient maps, where X_1 is formed from X_0 by identifying 0 and 1 (all rational points in X_0 with denominators 2^0), then form X_2 from X_1 by identifying 1/2 with the point 1-and-0 (all rational points with denominators 2^1 or less), then form X_3 by identifying all points with denominators 2^2 or less, and so on. Each space in the colimit is Hausdorff, yet one can anticipate that the *colimit* is the non-Hausdorff space of the previous example.

---

3. *Hausdorffness of quotients* G/H

The following important little result admonishes us further about dangers in taking quotients.

[3.0.1] Claim: Let G be a topological group, and H a subgroup. [19] The quotient topological space G/H is Hausdorff if and only if H is *closed*.

[3.0.2] Remark: Thus, even if H is *normal* in G, so that G/H is a group, if H is not closed then the quotient G/H does not meet the implicit Hausdorff-ness requirement of a topological group.

Proof: Suppose G/H is Hausdorff. Let $q : G \to G/H$ be the quotient map $q(g) = gH$. For $g \notin H$, $q(g) \neq q(1)$. By the Hausdorff-ness of the quotient, there are *disjoint* opens U, V in G/H such that $U \ni q(g)$ and $V \ni q(1)$. The inverse images $q^{-1}(U) \ni g$ and $q^{-1}(V) \supset H$ are still disjoint, and are open by continuity of q. Thus, $q^{-1}(U)$ is a neighborhood of g not meeting H. This holds for *every* $g \notin H$, so the complement to H is open, and H is closed.

For the more difficult converse, for H closed, given $x \notin H$, we will first find a neighborhood V of 1 in G such that

$$V \cdot x \cap V \cdot H = \phi$$

Then [20]

$$V \cdot xH \cap V \cdot H = \phi$$

This will imply that $q(V \cdot xH)$ and $q(V \cdot H)$ are *disjoint*. Observe that [21] for a subset X of G,

$$q^{-1}(q(X)) = X \cdot H = \{x \cdot h : x \in X,\ h \in H\}$$

Thus, $q(V \cdot xH)$ and $q(V \cdot H)$ will be *open*, [22] so will be disjoint neighborhoods of $q(x)$ and $q(1)$, respectively. And the general Hausdorff-ness will be reduced to this case.

[18] In terms of equivalence relations, to *identify* all points of a subset Y of a topological space X is to define an equivalence relation $\sim$ by saying that $y \sim y'$ for all $y, y' \in Y$, and otherwise $x \sim x'$ only for $x = x'$, and then take the quotient by this equivalence relation.

[19] Again, implicitly, a topological group is Hausdorff and locally compact.

[20] By right-multiplying by $h \in H$ and taking the union over all $h \in H$.

[21] From noting that, for $y \in G$ such that $q(y) \in q(X)$, we have $yH = xH$ for some $x \in H$, so $y \in xH \subset X \cdot H$.

[22] By construction of the quotient topology, a set Y in G/H is open if and only if $q^{-1}(Y)$ is open in G.

To find such V, use the local compactness to take a neighborhood U of 1, with compact closure $\overline{U}$. Then[23] $U \cdot x$ is a neighborhood of x, with closure $\overline{U} \cdot x$. Since $x \notin H$, for each $y \in \overline{U} \cdot x \cap H$, necessarily $y \neq x$. Thus, by Hausdorff-ness, there is an open neighborhood U_y of 1[24] and open neighborhood V_y of y such that

$$V_y \cap U_y \cdot x = \phi$$

Since $\overline{U} \cdot x \cap H$ is compact,[25] there is a finite list $y_1, \ldots, y_n$ of points in $\overline{U} \cdot x \cap H$ such that the V_{y_i} cover $\overline{U} \cdot x \cap H$. The *finite* intersection $W_o = \bigcap_i U_{y_i} \cdot x$ is open, and does not meet H. Then $W_o \cdot x^{-1}$ is a neighborhood of 1. Let V_o be an open neighborhood of 1 such that $V_o^2 \subset W_o x^{-1}$,[26] and let $V = V_o \cap V_o^{-1}$. We claim that

$$V \cdot x \cap V \cdot H = \phi$$

Indeed, if y were in this intersection, then for some $v \in V$

$$y \in V \cdot x \cap v \cdot H$$

Then

$$v^{-1} y \in v^{-1} \cdot V \cdot x \cap H \subset V \cdot V \cdot x \cap H \subset V_o^2 \cdot x \cap H \subset (W_o \cdot x^{-1}) \cdot x \cap H = W_o \cap H = \phi$$

contradiction. So $V \cdot x \cap V \cdot H = \phi$, as desired, and

$$V \cdot xH \cap V \cdot H = \phi$$

The general issue of Hausdorff-ness of G/H reduces to the previous discussion by moving opens around. Given $y, z \in G$ such that $yH \neq zH$, let $x = y^{-1} z$ and choose V as in the previous paragraph for this x. That is,

$$V \cdot y^{-1} zH \cap V \cdot H = \phi$$

Left multiply by y to get

$$yVy^{-1} \cdot zH \cap yV \cdot H = \phi$$

Rearrange slightly to have

$$yVy^{-1} \cdot zH \cap yVy^{-1} \cdot yH = \phi$$

Whatever else it may be, $W = yVy^{-1}$ is open in G and contains 1, so by our earlier observations, $q(W \cdot zH)$ and $q(W \cdot yH)$ are open, disjoint, and contain $q(zH)$ and $q(yH)$, respectively. This proves the Hausdorff-ness of G/H. ///

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4. *Ascending unions, strict colimits*

One lesson of the previous section is that general colimits may fail to have properties we need, such as Hausdorff-ness. Fortunately, our ascending union

$$\mathbb{Q}_2 = \bigcup_{n=1}^{\infty} 2^{-n} \mathbb{Z}_2 = \mathrm{colim}_n \, 2^{-n} \mathbb{Z}_2$$

---

[23] Since right multiplication by x is a homeomorphism of G to itself.

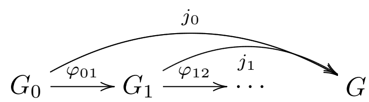
[24] As on many other occasions in this and similar discussions, from neighborhoods W of 1 we can make neighborhoods $W \cdot x$ of other points x by translating.

[25] This compactness results from H being closed and $\overline{U}$ being compact, since closed subsets of compacts are compact.

[26] That there is such a neighborhood V_o of 1 follows immediately from the continuity of $G \times G \to G$ by multiplication.

is a special sort of colimit.

A **strict colimit** is a colimit G of objects G_n where in the diagram



the maps $\varphi_{n,n+1} : G_n \to G_{n+1}$ are all *isomorphisms to their images.* [27]

[4.0.1] Remark: No, although we did prove that the $2^{-n}\mathbb{Z}_2$'s inject to their successors, we did not prove that these injections are isomorphisms to their images. [28] However, being bijective continuous maps from compact spaces to Hausdorff spaces, it *follows* that these inclusions are homeomorphisms. [29] Thus, $\mathbb{Q}_2$ is a *strict* colimit of $2^{-n}\mathbb{Z}_2$.

As usual, the mapping property definition specifies the object up to unique isomorphism, but it is useful to identify in greater detail the topology on a strict colimit.

[4.0.2] Claim: The **strict colimit** of topological spaces X_n (with inclusions $j_{n,n+1} : X_n \to X_{n+1}$) is the *set* strict colimit (ascending union) X (with inclusions $j_n : X_n \to X$) given the topology in which a set U in X is open if and only if each $X_n \cap U$ is open in X_n.

[4.0.3] Remark: Since the set strict colimit is the ascending union, we can safely reduce notational clutter by identifying each set X_n with its image in the ascending union X, and concommitantly identifying the set X_n with its image in X_{n+1}.

Proof: First, one should check that the sets U in X whose intersection $U \cap X_n$ with every X_n is open really do form a topology on X, but this is immediate. Further, this does show that the inclusions $X_n \to X_{n+1}$ are homeomorphisms to their images.

Given an open set U in X, its inverse image in X_n via j_n is simply $X_n \cap U$, which is open by definition of the topology on X. Thus, the inclusions $j_n : X_n \to X$ are continuous.

Given a compatible family of continuous maps $f_n : X_n \to Z$, define $f : X \to Z$ pointwise in the only way possible, namely

$$f(g) = f_n(g) \quad \text{(for any } n \text{ large enough so that } g \in X_n)$$

using the fact that the ascending union is the set-colimit. We certainly do have the compatibility

$$f(j_n(g_n)) = f_n(g_n)$$

at least as set maps, because the ascending union *is* a set-colimit. To prove *continuity* of f, let V be open in Z. Using the compatibility,

$$f^{-1}(V) \cap X_n = f_n^{-1}(V)$$

---

[27] The notion of *image* does not make sense in every category, although, luckily, it does have a sense in most familiar categories. Recall that a map $f : X \to Y$ is an *isomorphism to its image* if the map $f : X \to f(X)$ is an isomorphism, where $f(X)$ is the image of X in Y by f. If, for example, these are topological spaces, then $f(X)$ is given the subspace topology.

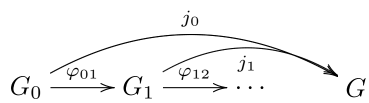
[28] Not every continuous bijection is a homeomorphism. As a stark example, mapping $\{0, 1\}$ to $\{0, 1\}$ by $0 \to 0$ and $1 \to 1$, where the source copy has the *discrete* topology (all subsets are open), while the target copy has the *indiscrete* topology (only the whole set and the empty set are open). This is continuous but not a homeomorphism.

[29] The proof of this useful fact is simple: it suffices to prove that the map takes opens to opens, or, equivalently, closed sets to closed. A closed subset of a compact set is compact, and the image of a compact set is compact. And then a compact subset of a Hausdorff space is closed, and we are done.

which is open in X_n by the assumed continuity of f_n. Thus, f is continuous. That is, the ascending union with this topology is a colimit of topological spaces. ///

[4.0.4] Remark: Note that we make no claim about Hausdorff-ness or local compactness without further hypotheses.

[4.0.5] Theorem: Let topological groups G_n fit into a *strict colimit* diagram



with each G_n *open* in G_{n+1}. Then the group colimit G is Hausdorff and locally compact, and the inclusions $G_n \to G$ are *open* maps. When the groups G_n act continuously on a topological space X, the strict colimit G acts continuously on X.

[4.0.6] Remark: Since the continuous maps $G_n \to G$ are *open* [30] they are homeomorphisms to their images. [31]

[4.0.7] Remark: To apply the theorem to the colimit $\mathbb{Z}_2 \to \frac{1}{2}\mathbb{Z}_2 \to \ldots$, the colimit must be *strict* and each image is *open* in the next object. The strictness was noted above.

Proof: We claim that the strict colimit topology on the ascending union of the G_n's (discussed in the previous claim) constructs a topological-group colimit. We identify G_n with its image in G_{n+1} and in the ascending union.

First, using the assumption that G_n is open in G_{n+1}, we prove that the inclusion $G_n \to G$ is an *open* map, that is, images of opens are open. Indeed, let U be open in G_n. By continuity of the transition maps, the inverse images of U in G_{n-1}, G_{n-2}, and so on are open. Since each inclusion $G_n \to G_{n+1}$ has open image and is a homeomorphism, the images of U in G_{n+1}, G_{n+2}, etc., are open. Thus, by definition of the topology, the image of U in G is open. In particular, G_n is open in G, and the inclusion $G_n \to G$ is a homeomorphism to its image.

For the Hausdorff-ness of G, for given $x \neq y \in G$, let n be large enough such that $x, y \in G_n$. Since G_n is Hausdorff, there are neighborhoods $U \ni x$ and $V \ni y$ in G_n such that $U \cap V = \phi$. Since U and V are still open in G, this gives the Hausdorff-ness of G.

To prove local compactness, given $g \in G$ again choose n large enough such that $g \in G_n$, and take a neighborhood U of g in G_n with compact closure. Since G_n is open in G, this neighborhood is a neighborhood of g in G as well, and has compact closure there, since the inclusion $G_n \to G$ is a homeomorphism to its image.

We give the ascending union a group structure compatible with those on the limitands. This is easy, since, given $x, y \in G$, for any large-enough index n we will have $x, y \in G_n$, and use the definition of the group operation in G_n. Since the maps $G_n \to G_{n+1}$ are group homomorphisms, we get the same answer regardless of the choice of n. Similarly, to prove associativity, given $x, y, z \in G$, choose n large enough such that $x, y, z \in G_n$, to infer $(xy)z = x(yz)$ inside G_n. The property of the identity, and existence of inverses follow similarly.

---

[30] Again, a map is open if it sends open sets to open sets.

[31] Again, for a map $f : A \to B$ to be a homeomorphism *to its image* means that the image $f(A)$ with the *subspace* topology inherited from the target space B is homeomorphic to A by $f : A \to f(A)$. And, again, this does *not* imply that $f : A \to B$ is a *surjection*.

Similarly, to get a group action of G on X, without worrying about topology, given $g \in G$ take n large enough such that $g \in G_n$, and use the definition of the action $g \cdot x$ for G_n. The compatibility of the actions of the various G_n's implies that this is well-defined. The associativity $g(g'x) = (gg')x$ follows similarly, as does $1 \cdot x = x$.

Next, in proving *continuity* of the group operation, given $x, y \in G$, take n large enough such that both x, y are in G_n. Then $x \cdot y \in G_n$. The group operation is continuous on $G_n \times G_n \to G_n$, and since G_n is open in G and the inclusion $G_n \to G$ is a homeomorphism to its image, this gives the continuity of $G \times G \to G$ at (x, y). The continuity of the inversion map is proven similarly from that on the individual G_n.

Similarly, to prove continuity of the action of G on X. Let $m : G \times X \to X$ be the action just defined, with $m_n : G_n \times X \to X$ the action of G_n. Let U be open in X. The compatibility of the inclusions with the actions m_n, together with the fact that G_n is open in G, implies that

$$m^{-1}(U) = \bigcup_n m_n^{-1}(U) = \text{union of opens} = \text{open}$$

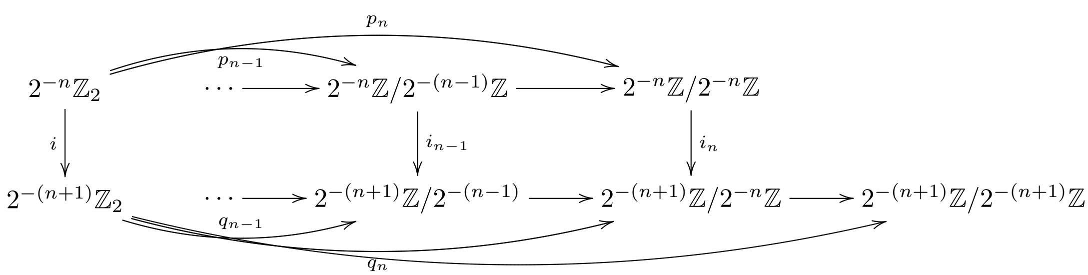
That is, the action of G on X is continuous. ///

To apply this theorem to

$$\mathbb{Q}_2 = \bigcup_{n=0}^{\infty} 2^{-n} \cdot \mathbb{Z}_2$$

what remains is to prove that each inclusion $2^{-n}\mathbb{Z}_2 \to 2^{-(n+1)}\mathbb{Z}_2$ has open image, since we have already observed that $\mathbb{Q}_2$ is *strict* as a colimit.

To this end, consider the diagram



To show that $i(2^{-n}\mathbb{Z}_2)$ is open in $2^{-(n+1)}\mathbb{Z}_2$, we will show that $i(2^{-n}\mathbb{Z}_2)$ is the inverse image $\ker q_n = q_n^{-1}(\{0\})$ of[32]

$$\{0\} = i_n(2^{-n}\mathbb{Z}/2^{-n}\mathbb{Z}) \subset 2^{-(n+1)}\mathbb{Z}/2^{-n}\mathbb{Z}$$

The projection q_n is continuous, so this inverse image will be open, as desired.

On one hand, the commutativity of the diagram shows immediately that

$$i(2^{-n}\mathbb{Z}_2) \subset \ker q_n$$

On the other hand, to prove equality in this containment, proceed as follows. The restrictions to $\ker q_n$ of all the projections q_ℓ have images equal to the images of the vertical isomorphisms-to-their-images i_ℓ, so we can create a compatible family of maps

$$f_\ell = i_\ell^{-1} \circ q_\ell : \ker q_n \to p_\ell(2^{-n}\mathbb{Z}_2)$$

[32] Each of the limitands is *finite*, so to be Hausdorff has no choice but to be given the discrete topology, the only Hausdorff topology on a finite set. In a discrete topology, any subset is open.

This induces a map

$$f : \ker q_n \to 2^{-n}\mathbb{Z}_2$$

compatible with all the maps f_ℓ. By now it is not surprising that f is a two-sided inverse to i. This will follow naturally from the compatibility

$$p_\ell \circ f = f_\ell = i_\ell^{-1} \circ q_\ell$$

combined with the compatibility

$$q_\ell \circ i = i_\ell \circ p_\ell$$

Indeed,

$$q_\ell \circ i \circ f = i_\ell \circ p_\ell \circ f = i_\ell \circ i_\ell^{-1} \circ q_\ell = q_\ell$$

As usual, only the identity map on a limit is compatible with all the projections, so

$$i \circ f = \text{ identity on } \ker q$$

And

$$p_\ell \circ f \circ i = i_\ell^{-1} \circ q_\ell \circ i = i_\ell^{-1} \circ i_\ell \circ p_\ell = p_\ell$$

so

$$f \circ i = \text{ identity on } 2^{-n}\mathbb{Z}_2$$

Thus, f and i are mutual inverses, so $i(2^{-n}\mathbb{Z}_2) = \ker q$, which is open in $2^{-(n+1)}\mathbb{Z}_2$.

This completes the verification of all the hypotheses for application of the theorem.

[4.0.8] Remark: Although it is possible to strengthen this discussion a bit, there are genuine complications in treatment of colimits of topological groups, in contrast to limits.
