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The ur-solenoid and the adeles

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Most of the discussion of the 2-solenoid did not depend upon the 2, apart from the fact that 2 is an integer larger than 1. We consider a larger family of solenoids, at first with 2 replaced by a larger integer N, suggesting that such limits *factor* according to the prime factorization of N. This factorization does occur, and requires a proof which can be given in a general context.

Incidentally, to make the *universal* ur-solenoid, we must also improve our indexing of limits, and understand the effect of variation of indexing on the *limit object*.

Finally, we do make the ur-solenoid (in notation below)

$$\lim_n \mathbb{R}/n\mathbb{Z} \approx (\mathbb{R} \times \lim_n \mathbb{Z}/n)/\mathbb{Z}^\Delta \approx (\mathbb{R} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \ldots)/\mathbb{Z}^\Delta$$

where the product is over p-adic integers $\mathbb{Z}_p$ for p prime. This introduces

$$\widehat{\mathbb{Z}} = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11} \times \ldots \ = \ \Pi_{p \text{ prime}} \mathbb{Z}_p$$

Then we have the **finite adeles**

$$\text{finite adeles} = \mathbb{A}_{\text{fin}} = \text{colim}_n \frac{1}{n} \cdot \widehat{\mathbb{Z}}$$

As usual, we absorb the factor of $\mathbb{R}$, defining the full **adeles**

$$\text{adeles} = \mathbb{A} = \mathbb{R} \times \mathbb{A}_{\text{fin}} \ = \ \text{colim}_n \frac{1}{n} \cdot (\mathbb{R} \times \widehat{\mathbb{Z}})$$

and our final expression for the ur-solenoid is

$$\text{ur-solenoid} \ = \ (\mathbb{R} \times \mathbb{A}_{\text{fin}})/\mathbb{Q}^\Delta = \mathbb{A}/\mathbb{Q}^\Delta$$

Yes, as this expression and our previous discussion suggest, the full abelian group of rational numbers is *discrete* in $\mathbb{A}$, and this quotient is *compact*.

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1. *Wider solenoids*

By the same arguments as for the 2-solenoid, we obtain

[1.0.1] Claim: For any integer $N > 1$, the N-solenoid, defined as the limit of the projective system

$$\cdots \xrightarrow{\text{mod } N^n} \mathbb{R}/N^n\mathbb{Z} \xrightarrow{\text{mod } N^{n-1}} \mathbb{R}/N^{n-1}\mathbb{Z} \xrightarrow{\text{mod } N^{n-2}} \cdots \xrightarrow{\text{mod } N} \mathbb{R}/N\mathbb{Z} \xrightarrow{\text{mod } 1} \mathbb{R}/\mathbb{Z}$$

is isomorphic to

$$(\mathbb{R} \times \lim \mathbb{Z}/N^n) \ / \ \mathbb{Z}^\Delta$$

as a $\mathbb{R} \times \lim \mathbb{Z}/N^n$ space,[1] where $\lim \mathbb{Z}/N^n$ refers to the projective limit (topological group) of the system

$$\ldots \xrightarrow{\text{mod } N^n} \mathbb{Z}/N^n \xrightarrow{\text{mod } N^{n-1}} \mathbb{Z}/N^{n-1} \xrightarrow{\text{mod } N^{n-2}} \ldots \xrightarrow{\text{mod } N} \mathbb{Z}/N \xrightarrow{\text{mod } 1} \mathbb{Z}/\mathbb{Z}$$

(without further claims, for the moment, about the nature of $\lim \mathbb{Z}/N^n$, other than that it is a compact topological group, since it is the projective limit of finite, hence compact, topological groups). ///

We should surely recall that, for $N = ab$ a *composite* number[2] with a and b relatively prime,[3] *Sun-Ze's* theorem[4] gives an isomorphism of rings, so certainly of abelian groups,

$$\mathbb{Z}/N \approx \mathbb{Z}/a \times \mathbb{Z}/b$$

For example, for any exponent n,

$$\mathbb{Z}/6^n \approx \mathbb{Z}/2^n \times \mathbb{Z}/3^n$$

Thus, optimistically, Sun-Ze's theorem suggests that a limit $\lim \mathbb{Z}/N^n$ could be expressed as a *product* of *smaller* limits.[5]

[1.0.2] Claim: Let $N = p_1^{e_1} \ldots p_t^{e_t}$ be a factorization of N into powers of distinct primes p_i. Then

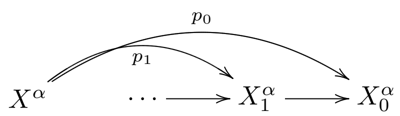
$$\lim_n \mathbb{Z}/N^n \approx \Pi_{i=1}^t \mathbb{Z}_{p_i}$$

where for prime p,[6]

$$\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$$

This claim will be a corollary of a much more general result, proven in the next section, namely

[1.0.3] Claim: Limits of products are products of limits. That is, given a *family* of limits



with α in an index set A, we have

$$\Pi_\alpha \left(\lim_n X_n^\alpha \right) \approx \lim_n \left(\Pi_\alpha X_n^\alpha \right)$$

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[1] Recall that, for a topological group G, a G-space is a topological space A with a *continuous* action of G on it. A map $f : A \to B$ of G-spaces is a continuous map of G spaces respecting the action(s) of G in the sense that $f(g \cdot a) = g \cdot f(a)$.

[2] Here *composite* means a non-prime integer larger than 1.

[3] As usual, two integers are *relatively prime* or *coprime* if their greatest common divisor is 1.

[4] This result is sometimes called the *Chinese remainder theorem*, but this name is inappropriate, for more than one reason.

[5] Just on the grounds that we seem unwilling to write $\mathbb{Z}_N$ for that projective limit, *unlike* our willingness to write $\mathbb{Z}_2$ for the $N = 2$ case, the clever reader could speculate that the case of general N is not the same as the case $N = 2$, without having any other reason in mind.

[6] And for *prime* p the limit $\lim_n \mathbb{Z}/p^n$ is the p-adic integers, which we will verify by comparison to the usual definition shortly.

(Proof in next section)

A trickier example is the limit

$$\text{factorial solenoid} \;=\; \lim_n \mathbb{R}/n! \cdot \mathbb{Z}$$

which, again by the same arguments as with the 2-solenoid, is isomorphic as $\mathbb{R} \times \lim_n \mathbb{R}/n!$ space to

$$\lim_n \mathbb{R}/n! \cdot \mathbb{Z} \approx (\mathbb{R} \times \lim_n \mathbb{Z}/n!)/\mathbb{Z}^\Delta$$

In this case, $\lim_n \mathbb{Z}/n!$ still might seem as though it should factor, according to the prime-power factors of $n!$, from Sun-Ze's theorem. That is, for example, of course,

$$\mathbb{Z}/9! = \mathbb{Z}/362880 = \mathbb{Z}/2^7 \cdot 3^4 \cdot 5 \cdot 7 \approx \mathbb{Z}/2^7 \times \mathbb{Z}/3^4 \times \mathbb{Z}/5 \times \mathbb{Z}/7$$

Indeed, given an integer N, for sufficiently large n the factorial $n!$ is divisible by N, so eventually every $\mathbb{Z}/N$ appears as a *quotient* of some $\mathbb{Z}/n!$. But the precise description of the pattern is messy, whether or not one can invent a notational system sufficient to write a formula. Thus, it is less obvious, still plausible, and in fact true, that

$$\lim_n \mathbb{Z}/n! \;\approx\; \Pi_{p \text{ prime}} \lim_e \mathbb{Z}/p_e \;=\; \Pi_{p \text{ prime}} \mathbb{Z}_p$$

To understand this without becoming embroiled in notational churn will require a revision of our viewpoint on limits, after discussion of *limits of products*.

In fact, we will make a more symmetrical presentation of a *universal solenoid* after preparation concerning the wider possibilities for indexing objects in a limit.

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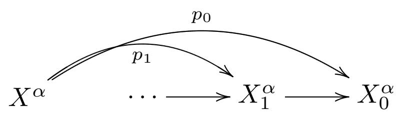
2. *Products of limits, limits of products*

Again, we certainly hope that plausible isomorphisms such as

$$\lim_n \mathbb{Z}/6^n \approx \mathbb{Z}_2 \times \mathbb{Z}_3$$

really do hold, and for robust, general reasons. Happily, this is so:

[2.0.1] Claim: Limits of products are products of limits. That is, given a *family* of limits



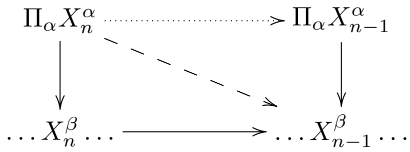
with α in an index set A, we have

$$\Pi_\alpha \left(\lim_n X_n^\alpha\right) \approx \lim_n \left(\Pi_\alpha X_n^\alpha\right)$$

[2.0.2] Remark: The projection maps from $\Pi_\alpha(\lim_n X_n^\alpha)$ to the limits $\lim_n X_n^\alpha$ are the usual things. It is less immediate what the maps from $\lim_n(\Pi_\alpha X_n^\alpha)$ to the individual products $\Pi_\alpha X_n^\alpha$ should be, since we must have transition maps

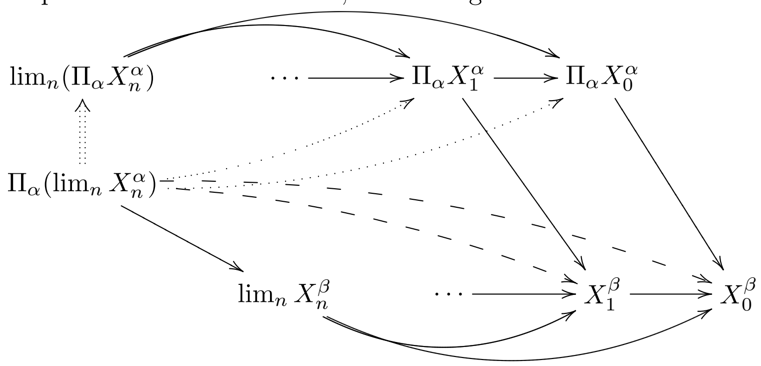
$$\Pi_\alpha X_n^\alpha \to \Pi_\alpha X_{n-1}^\alpha$$

In fact, these are induced via



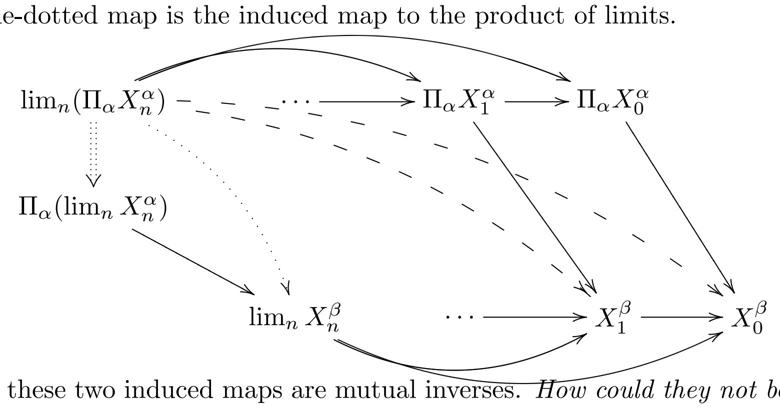
where the solid arrows are the natural projections or transition maps, the dashed arrows are the composites, and the dotted arrow is induced via the defining property of the product $\Pi_\alpha X_{n-1}^\alpha$. It is also easy to write formulas for the maps using the Cartesian product construction for products.

Proof: First, to get a map from the product of the limits to the limit of the products, keep in mind that to give a map to a product is to give a map to all the factors, and to give a map to a limit is to give a compatible family of maps to the limitands. Then, in the diagram



all the solid arrows are the natural maps we already have, the dashed arrows are the obvious composites, the dotted arrows are a compatible family induced to the products from maps to the factors, and the double-dotted map is induced to the limit from the single-dotted compatible maps. This gives a map (the double-dotted one) from the product of limits to the limit of the products.

To get the map in the other direction, the roles are essentially reversed, with minor modifications due to the fact that in a product, unlike limits, there are no compatibility requirements among the projections. [7] Thus, in the following diagram, the dashed lines are composites, the dotted map is the induced map to the limit, and the double-dotted map is the induced map to the product of limits.



We must claim that these two induced maps are mutual inverses. *How could they not be?* Indeed, we *should* talk ourselves into believing that the conclusion is nearly obvious. However, there is the issue of giving a genuine proof that also communicates effectively. The *idea* is that we work our way back up or down the diagram, essentially repeating the arguments already given. Then as usual invoke the fact that limits and products have no maps to themselves other than the identity that commute with all projections. ///

3. *Cofinal limits are isomorphic*

We must allow ourselves a more flexible, general form of indexing objects, and maps among them, in a projective limit. The best presentation of the *universal solenoid* already requires this, as we will see afterward.

[7] In fact, *products* and the sort of *limits* we have been considering are two of the more interesting instances of a more general construction, whose generality is mitigated by notational clumsiness. If one feels compelled to highlight the symmetry, one should address the puzzle of symmetrizing the diagram.

Especially in the context of more complicated indexing sets for limits, it is useful and reasonable to imagine that superficial changes in the indexing of a limit ought *not* to affect the limit object itself. [8]

For example, rather than the sequence of integers

$$\ldots,\ 2^4,\ 2^3,\ 2^2,\ 2,\ 1$$

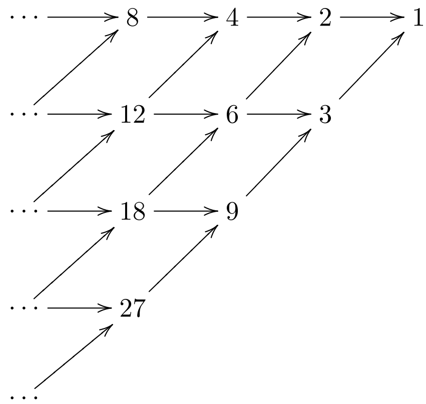
or obvious generalizations

$$\ldots,\ N^4,\ N^3,\ N^2,\ N,\ 1$$

mentioned just above, we will want to look at various more complicated **posets** [9] of positive integers, in each case **ordered by divisibility** in the sense that

$$n > m \quad \Longleftrightarrow \quad m \text{ divides } n$$

In particular, we no longer insist that any two integers in the set are comparable in the divisibility order. [10] For example, taking all integers which are products of powers of 2 and 3, we make a picture by drawing $n \to m$ for $n > m$:



and we suppress arrows that are composites, meaning that if $a > b > c$ then we do *not* draw an arrow from a to c. (Otherwise the visual clutter is terrible.) [11]

Given a poset I of positive integers, we might have objects X_n and transition maps

$$\varphi_{nm} : X_n \to X_m \quad \text{for} \quad n > m$$

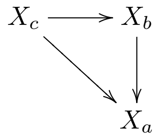
[8] This behavior is reminiscent of the fact that a subsequence of a convergent sequence necessarily has the same limit as the original sequence. Of course, this fact about sequences does not use a complicated indexing of the objects.

[9] The coinage *poset* is a not-too-bad-sounding contraction of *partially ordered set*, which has the following standard sense. A partial order $<$ on a set is a binary relation such that: if $x < y$ then $x \not< y$ (antisymmetry, which implies that $x \not< x$); if $x < y$ and $x < z$ then $x < z$ (transitivity). Note that we do *not* require any sort of dichotomy or trichotomy, that is, we do *not* require that either $x < y$, $x > y$, or $x = y$. If the latter condition *does* hold then the relation $<$ is a *total* order. A set with a partial order is a *poset.* A set with a total order is *not* a *toset*, but just an *ordered set.*

[10] In the jargon of ordered sets, we no longer have have a *total* ordering (wherein any two elements are comparable), but only a *partial* ordering.

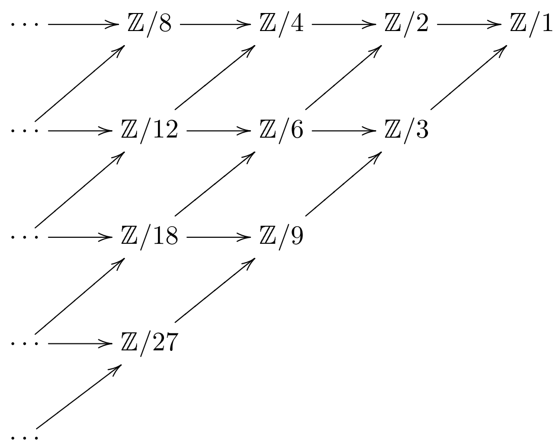
[11] There is a related notion of **net** inside a topological space, which means an image of a poset in that topological space, just as a *sequence* is an image of the poset of positive (or non-negative) integers with the usual ordering by magnitude.

which are *compatible* in the expected sense that all triangles commute (suppressing the names for the transition maps)



(for $c > b > a$ in I)

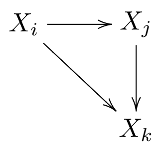
A simple meaningful example of a set naturally indexed in this manner has objects indexed by integers of the form $2^s 3^t$ ordered by divisibility



where, as expected, $\mathbb{Z}/n \to \mathbb{Z}/m$ for $m|n$ is reduction modulo m.

Such diagrams begin to accidentally achieve as much meaningless complexity as capturing useful information. This motivates setting up a stronger language for this sort of discussion.

Again, a **poset** is a *partially ordered set*, meaning that there is an order relation which is anti-reflexive, anti-symmetric, and transitive, *without* the assumption that any two elements are comparable. A **projective system** indexed by a poset I consists of objects X_i for all $i \in I$, and *transition maps* $\varphi_{ij} : X_i \to X_j$ for all i, j in I with $i > j$, which are required to be *compatible* in the sense that for $i > j > k$ in I, we have a commutative triangle (of transition maps, with labels suppressed)

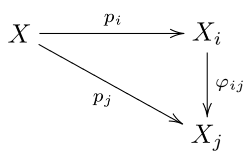


(for $i > j > k$ in I)

A (projective) **limit** of this system is an object X and *projections* $p_i : X \to X_i$ for all $i \in I$, *compatible with* the transition maps φ_{ij} for $i > j$ in the sense that

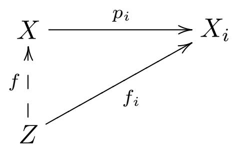
$$p_j = \varphi_{ij} \circ p_i$$

That is, we have commutative triangles



(for $i > j$ in I)

And for any *other* object Z with compatible maps $f_i : Z \to X_i$ (compatibility meaning that $f_j = \varphi_{ij} \circ f_i$ for all $i > j$) there is a unique $f : Z \to X$ through which all the maps f_i factor, in the sense that that for all i we have a commutative triangle



(for all $i \in I$)

We write

$$\lim_{i \in I} X_i = X$$

as usual suppressing reference to the transition maps and projection maps.

[3.0.1] Remark: Apart from allowing more complicated indexing schemes, this is just the notion of limit we've already used.

A poset is **directed** if, for two indices $i, j \in I$, there is $k \in I$ such that $k > i$ and $k > j$. Most index sets we'll consider for limits do have this property.

A subset J of a directed set I is **cofinal** in I if, for every $i \in I$, there is $j \in J$ with $j \geq i$. Two subsets J, J' of a poset I are (mutually) **cofinal** if, for each $j \in J$ there is $j' \in J'$ with $j' \geq j$, and for each $j' \in J'$ there is $j \in J$ with $j \geq j'$. Roughly, neither quits before the other.

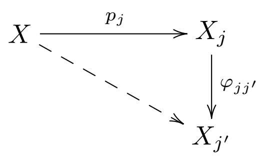
[3.0.2] Theorem: Let J and J' be mutually cofinal *directed* subsets of a poset I, and let $\{X_i : i \in I\}$ be a projective system. Then there is a *natural* isomorphism

$$\lim_{i \in J} X_i \approx \lim_{i \in J'} X_i$$

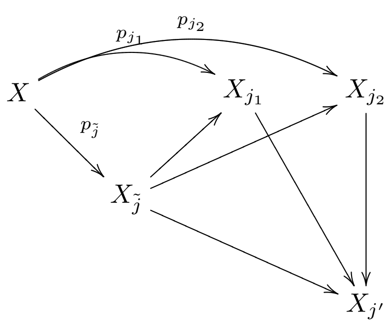
compatible with all the transition maps.

[3.0.3] Remark: The *directedness* does play a definite role in the proof. It is not just a matter of avoiding invocation of the Axiom of Choice (which itself would not be so bad).

Proof: Let $X = \lim_{i \in J} X_i$ and $X' = \lim_{i \in J'} X_i$. To make a map $\alpha : X \to X'$ is to make a compatible family of maps $X \to X_{j'}$ for all $j' \in J'$. Using the cofinality, for given $j' \in J'$ choose $j \in J$ with $j \geq j'$, and define $X \to X_{j'}$ as the (dotted) composite

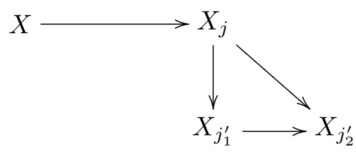


The next point is to be sure that these maps are *independent* of the choice of $j \in J$ with $j \geq j'$. Let j_1 and j_2 be in J with $j_1 \geq j'$ and $j_2 \geq j'$. Using the *directedness* of J, let $\tilde{j} \in J$ be such that $\tilde{j} \geq j_1$ and $\tilde{j} \geq j_2$. All the triangles commute in the diagram

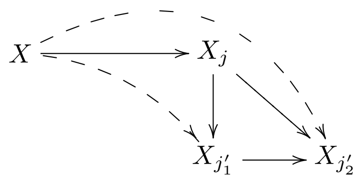


so the composites from X through X_{j_1} and X_{j_2} are the same, since they are equal to the composite through $X_{\tilde{j}}$.

Next, we prove that these maps $X \to X_{j'}$ are *compatible*. Let $j'_1 \geq j'_2$ in J', and take $j \in J$ such that $j \geq j'_1$. By transitivity, $j \geq j'_2$. First, we have a commutative diagram

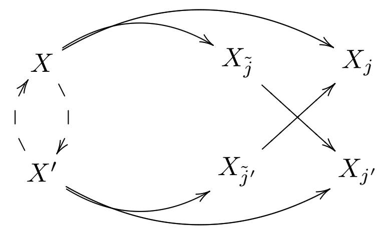


which give dashed composite maps and a commutative diagram



which proves that the maps $X \to X_{j'}$ for $j' \in J'$ are compatible. Thus, finally, we do have an induced map from $X = \lim_{i \in J} X_i$ to $X' = \lim_{i \in J'} X_i$.

The same argument, reversed, gives an induced map in the other direction. As usual, we should anticipate that these two maps are mutual inverses, from which the isomorphism of X and X' follows. The details of the proof of this are akin to the proof that any two limits are naturally isomorphic, with notational complications. Thus, for given $j \in J$ and $j' \in J'$, take $\tilde{j} \in J$ with $\tilde{j} \geq j'$ and $\tilde{j}' \in J'$ with $\tilde{j}' \geq j$. The cofinality assures that this is possible. We have commutative diagrams with the dotted maps being induced as above



By construction, the composite $X \to X' \to X_j$ (for $j \in J$) is the same as the projection $X \to X_j$, so the induced $X \to X' \to X$ is a self-map on X that respects the projections to the X_j ($j \in J$), so must be the identity on X. Symmetrically, the other composite $X' \to X \to X'$ is the identity on X', so both maps are isomorphisms. ///

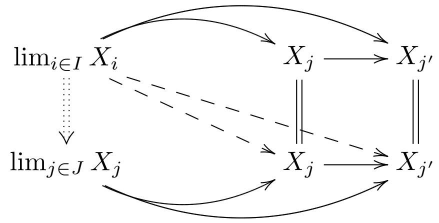
[3.0.4] Remark: The fact that cofinal limits are natually isomorphic sometimes arises as an issue of *well-definedness.*

4. *The ur-solenoid and the adeles*

Generally, for a subset J of a poset I, there is a unique natural map

$$\lim_{i \in I} X_i \to \lim_{j \in J} X_j$$

from the bigger limit to the smaller, giving a commutative diagram



(for all $j, j' \in J$)

with the dashed lines really none other than projections, though to a subset of all the limitands of the full system, then inducing a unique map from the larger to the smaller limit. That is, the smaller limit is a *quotient* of the larger limit.

Thus, all the various sub-posets

$$\begin{array}{ccccccccccc} \ldots & > & 16 & > & 8 & > & 4 & > & 2 & > & 1 \\ \ldots & > & 81 & > & 27 & > & 9 & > & 3 & > & 1 \\ \ldots & > & 1296 & > & 216 & > & 36 & > & 6 & > & 1 \\ \ldots & > & 5! & > & 4! & > & 3! & > & 2! & > & 1! \end{array}$$

of positive integers *ordered by divisibility* [12] give limits which are *quotients* of the limit over *all* positive integers ordered by divisibility.

Because the thing arises so often, the product over all *primes* of the p-adic integers

$$\mathbb{Z}_p = \lim_\ell \mathbb{Z}/p^\ell$$

has a standard notation

$$\widehat{\mathbb{Z}} = \Pi_{p \text{ prime}} \mathbb{Z}_p$$

Further, the colimit of $\frac{1}{n}\widehat{\mathbb{Z}}$ arises very often, is called the **finite adeles**, and denoted

$$\text{finite adeles} = \mathbb{A}_{\text{fin}} = \text{colim}_n \frac{1}{n} \cdot \widehat{\mathbb{Z}}$$

Often the finite adeles occur with a further factor of $\mathbb{R}$, and the product of the finite adeles with $\mathbb{R}$ is the (full) **adeles** [13]

$$\text{adeles} = \mathbb{A} = \mathbb{R} \times \mathbb{A}_{\text{fin}}$$

[4.0.1] Theorem: The limit $\lim_n \mathbb{R}/n\mathbb{Z}$ over all positive integers, ordered by divisibility, of the circles $\mathbb{R}/n\mathbb{Z}$ with transition maps

$$\mathbb{R}/n\mathbb{Z} \to \mathbb{R}/m\mathbb{Z} \qquad (\text{for } m|n)$$

as an $\mathbb{R} \times \widehat{\mathbb{Z}}$ space is

$$\lim_n \mathbb{R}/n\mathbb{Z} \approx (\mathbb{R} \times \widehat{\mathbb{Z}})/\mathbb{Z}^\Delta$$

In fact, this limit is an $\mathbb{A}$-space

$$\lim_n \mathbb{R}/n\mathbb{Z} \approx \mathbb{A}/\mathbb{Q}^\Delta$$

and the copy of $\mathbb{Q}$ is *discrete* in $\mathbb{A}$.

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[12] In fact, we will see that the factorial solenoid *is* already the universal one, but presenting it as such would make determining its *automorphisms* relatively awkward. The present more balanced approach is much more informative.

[13] Actually, these adeles are the *rational* adeles, meaning the adeles attached to the rational numbers $\mathbb{Q}$. Each finite field extension k of $\mathbb{Q}$ has its own adeles, as well, sometimes denoted $\mathbb{A}_k$, and obtained as $\mathbb{A}_k = \mathbb{A} \otimes_{\mathbb{Q}} k$, although the sense of this tensor product requires considerable explanation.

Proof: The same argument used to see that

$$\lim_\ell \mathbb{R}/2^\ell\mathbb{Z} \approx (\mathbb{R} \times \lim_\ell \mathbb{Z}/2^\ell)/\mathbb{Z}^\Delta$$

shows that

$$\lim_n \mathbb{R}/n\mathbb{Z} \approx (\mathbb{R} \times \lim_n \mathbb{Z}/n)/\mathbb{Z}^\Delta$$

To break the limit of $\mathbb{Z}/n$'s into nicer pieces, we first invoke Sun-Ze's theorem that

$$\mathbb{Z}/n \approx \Pi_{p \text{ prime}}\mathbb{Z}/p^e$$

where $n = \Pi p^e$ is the factorization of n into prime powers p^e. We will need a little more detail, so we introduce the standard (if clunky) notation

$$\text{ord}_p n = \text{ largest positive integer } e \text{ such that } p^e|n$$

Thus, generally, for $0 < n \in \mathbb{Z}$,

$$n = \Pi_{p \text{ prime}} p^{\text{ord}_p n}$$

Then

$$\lim_n \mathbb{Z}/n = \lim_n \left(\Pi_p \mathbb{Z}/p^{\text{ord}_p n}\right) \approx \Pi_p \left(\lim_n \mathbb{Z}/p^{\text{ord}_p n}\right)$$

For fixed prime p, we *expect* to show that

$$\lim_n \mathbb{Z}/p^{\text{ord}_p n} \approx \mathbb{Z}_p$$

However, the way that the exponent e depends upon n is irregular, *and* the poset of positive integers with divisibility is certainly *not* isomorphic to

$$\ldots > p^4 > p^3 > p^2 > p > 1$$

But by now we have proven that cofinal limits are isomorphic to the original limit, so a *proof* of the expected isomorphism can be obtained from taking a limit over a conveniently chosen cofinal (under divisibility) set of positive integers. One choice of subset is the set of factorials

$$B = \{1!,\ 2!,\ 3!,\ 4!,\ \ldots\}$$

This is cofinal since, given an integer n, there is ℓ such that $n|\ell!$. Thus,

$$\lim_n \mathbb{Z}/n \approx \lim_\ell \mathbb{Z}/\ell! \approx \lim_\ell \left(\Pi_p \mathbb{Z}/p^{\text{ord}_p \ell!}\right) \approx \Pi_p \left(\lim_\ell \mathbb{Z}/p^{\text{ord}_p \ell!}\right)$$

Next, for a fixed prime p, we will choose a subsequence of integers such that the powers of p dividing their factorials *strictly increase.* That is, for positive integer N and prime p, let $e = \text{ord}_p N$ be the maximal integer such that $p^e|N$. Then, for *fixed* prime p, choose a subsequence $\ell_1, \ell_2, \ldots$ of positive integers such that

$$\text{ord}_p \ell_1! < \text{ord}_p \ell_1! < \text{ord}_p \ell_2! < \ldots$$

Then, since cofinal limits are isomorphic,

$$\lim_n \mathbb{Z}/p^{\text{ord}_p n} \approx \lim_\ell \mathbb{Z}/p^{\text{ord}_p \ell!} \approx \lim_i \mathbb{Z}/p^{\text{ord}_p \ell_i!}$$

But the last limit is *also* cofinal in

$$\lim_k \mathbb{Z}/p^k \approx \mathbb{Z}_p$$

Thus, using the behavior of products under limits and the isomorphism of cofinal limits, we do have

$$\lim_n \mathbb{Z}/n = \lim_n \left(\Pi_p \mathbb{Z}/p^{\mathrm{ord}_p n}\right) \approx \Pi_p \left(\lim_n \mathbb{Z}/p^{\mathrm{ord}_p n}\right) \approx \Pi_p \mathbb{Z}_p = \widehat{\mathbb{Z}}$$

As with the 2-solenoid, there is the usual factor of $\mathbb{R}$ acting, as well. That the isotropy subgroup of the point 0 in the ur-solenoid is a diagonal copy of $\mathbb{Z}$ follows for the same reason as for the 2-solenoid. This is the first part of the theorem.

Recall that expressing the 2-solenoid as a limit of a *larger* diagram

$$\ldots \to \mathbb{R}/4\mathbb{Z} \to \mathbb{R}/2\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\frac{1}{2}\mathbb{Z} \to \mathbb{R}/\frac{1}{4}\mathbb{Z} \to \ldots$$

made visible a larger group of automorphisms

$$\mathbb{R} \times \mathrm{colim}_m \frac{1}{2^m}\,\mathbb{Z}_2 = \mathbb{R} \times \mathbb{Q}_2$$

Similarly, for the ur-solenoid we can consider the larger family of objects

$$\mathbb{R}/\frac{m}{n}\mathbb{Z} \quad (m \text{ and } n \text{ integers, } n \neq 0)$$

with obvious transition maps

$$\mathbb{R}/\frac{M}{N}\mathbb{Z} \to \mathbb{R}/\frac{m}{n}\mathbb{Z} \quad (\text{when there is } \ell \in \mathbb{Z} \text{ with } \tfrac{m}{n} \cdot \ell = \tfrac{M}{N})$$

By now invocation of the isomorphism of cofinal limits immediately assures us that the limit is the same. [14] Indeed, for fixed denominator n, the resulting subsystem of objects

$$\lim_m \mathbb{R}/\frac{m}{n}\mathbb{Z} \qquad (\text{positive integers } m, \text{ ordered by divisibility})$$

still has the same limit. And *now*, as with the 2-solenoid, we see that the group

$$\lim_m \frac{1}{n}\mathbb{Z}/\frac{m}{n} \cdot \mathbb{Z} \approx \frac{1}{n} \cdot \lim_m \mathbb{Z}/m = \frac{1}{n} \cdot \widehat{\mathbb{Z}}$$

acts on the ur-solenoid. Then the ascending union (colimit)

$$\mathbb{A}_{\mathrm{fin}} = \mathrm{colim}_n \frac{1}{n} \cdot \widehat{\mathbb{Z}} \qquad\qquad (\text{with } \tfrac{1}{m}\widehat{\mathbb{Z}} \text{ closed in } \tfrac{1}{n}\widehat{\mathbb{Z}} \text{ for } m|n)$$

acts. With the usual factor of $\mathbb{R}$ also acting, we do find that

$$\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$$

acts on the ur-solenoid.

Last, we verify that the isotropy group of the point 0 in the ur-solenoid is a diagonal copy $\mathbb{Q}^{\Delta}$ of the rational numbers $\mathbb{Q}$, and that this copy of $\mathbb{Q}$ is *discrete* in $\mathbb{A}$.

Suppose that $r \times x \in \mathbb{R} \times \frac{1}{n} \lim_m \mathbb{Z}/m$ fixes 0 in the universal solenoid. Use the model of $\frac{1}{n}\widehat{\mathbb{Z}} = \frac{1}{n} \lim_m \mathbb{Z}/m$ as a collection x of $x_m \in \frac{1}{n} \cdot \mathbb{Z}$ such that for all $m|M$ we have $x_M + \frac{m}{n}\mathbb{Z} = x_m + \frac{m}{n}\mathbb{Z}$. Then

$$r + x_m \in \frac{1}{n} \cdot m\mathbb{Z}$$

[14] In the earlier episode regarding the 2-solenoid, we gave a more pedestrian and direct argument, but which needlessly involved details of the situation.

for all m, so $r \in \frac{1}{n}\mathbb{Z}$, not merely in $\mathbb{R}$. Since

$$x_m + \frac{m}{n}\mathbb{Z} = -r + \frac{m}{n}\mathbb{Z}$$

and since $x = \{x_m\} \in \frac{1}{n}\widehat{\mathbb{Z}}$ is determined completely by just the cosets $x_m + \frac{m}{n}\mathbb{Z}$, we see that for $r \times x$ in the isotropy group

$$r \times x = r \times (-r) \in \frac{1}{n}\mathbb{Z}^\Delta$$

as claimed. Taking the union over $1 \le n \in \mathbb{Z}$ gives $\mathbb{Q}$ imbedded diagonally in $\mathbb{R} \times \mathbb{A}_{\text{fin}}$.

To prove the *discreteness* [15] of the isotropy group $\mathbb{Q}^\Delta$ in the adeles $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\text{fin}}$, we observe that it suffices to prove that the identity element in $\mathbb{Q}^\Delta$ is *isolated* in $\mathbb{A}$. [16] We know by now that $\mathbb{R} \times \widehat{\mathbb{Z}}$ is open in the colimit $\mathbb{R} \times \mathbb{A}_{\text{fin}}$, and contains 0. And

$$(\mathbb{R} \times \widehat{\mathbb{Z}}) \cap \mathbb{Q}^\Delta = \text{isotropy group of 0 in } \mathbb{R} \times \widehat{\mathbb{Z}} = \mathbb{Z}^\Delta$$

In fact, $\mathbb{Z}$ is already discrete in $\mathbb{R}$, so $\mathbb{Z}^\Delta$ is inevitably discrete in $\mathbb{R} \times \widehat{\mathbb{Z}}$, and then $\mathbb{Q}^\Delta$ is discrete in $\mathbb{R} \times \mathbb{A}_{\text{fin}}$. ///

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[15] Recall that a subset Γ of a topological space G is *discrete* if each element $\gamma \in \Gamma$ has a neighborhood U such that $U \cap \Gamma = \{\gamma\}$.

[16] This makes use of the fact that the subset to be proven discrete is a *subgroup* of a topological group. Suppose that there is an open neighborhood U of 1 in a topological group G such that $U \cap \Gamma = \{1\}$. For $\gamma \in \Gamma$, the map $x \to \gamma x$ is a homeomorphism (as discussed earlier) from U to a neighborhood γU of γ, and $\gamma U \cap \Gamma = \gamma(U \cap \gamma^{-1}\Gamma)$. Since Γ is a group, $\gamma^{-1}\Gamma = \Gamma$, and $\gamma(U \cap \Gamma) = \gamma \cdot \{1\} = \{\gamma\}$ as claimed.