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Toward Waveforms

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Waveforms are automorphic forms [1] that arise naturally in the harmonic analysis of quotients like $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, and were first studied in the late 1940's by Hans Maaß. These are subtler than the *holomorphic* elliptic modular forms on $\mathfrak{H}$ that arose in the context of elliptic functions over $\mathbb{C}$, in the mid-19^{th} century, but are more natural from the viewpoint of harmonic analysis. And waveforms illustrate techniques that are relevant beyond the classical methods from complex analysis (although complex-analysis methods in new incarnations are just as important as ever).

The harmonic-analysis goal that shapes our discussion is that of expressing every reasonable function f on the upper half-plane $\mathfrak{H}$ as a sum and/or integral of $\Delta^{\mathfrak{H}}$-eigenfunctions, with the usual $SL_2(\mathbb{R})$-invariant Laplacian

$$\Delta^{\mathfrak{H}} \;=\; y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$

The group $\Gamma = SL_2(\mathbb{Z})$ can be replaced by smaller congruence subgroups Γ, but this complication adds little interest and adds several unhelpful *secondary* complications.

The name *waveform* is a historical artifact, refering to a $\mathbb{C}$-valued Γ-invariant eigenfunction for $\Delta^{\mathfrak{H}}$, with *moderate growth* conditions to exclude phenomena that are usually undesirable. Perhaps the name is effectively a contraction of *wave-like automorphic form*.

The easiest and most explicit examples of waveforms are *Eisenstein series*, an example of which is

$$E_s(z) \;=\; \tfrac{1}{2}\sum_{c,d}\frac{y^s}{|cz+d|^{2s}} \qquad\qquad (c, d \text{ coprime integers, } \mathrm{Re}(s) > 1)$$

Every other example is significantly more complicated to describe. Although they play a critical role in the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$, Eisenstein series themselves are not L^2, as we will see in detail below. By contrast, L^2 waveforms are not well understood. An exception is the family of *special waveforms* found by Maaß, described quite a bit later in a better technical set-up. To prove (as did Selberg) that there are infinitely many *square-integrable* waveforms for $SL_2(\mathbb{Z})$ requires a *trace formula*, a sophisticated set-up with many technical pitfalls. The Maaß special waveforms are not $SL_2(\mathbb{Z})$-invariant, but only invariant under smaller congruence subgroups.

(For a while, people speculated that there might be some connection between the eigenvalues of the Laplacian on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ and the zeros of the zeta function. Since no alternative non-numerical characterization was known for the eigenvalues, it was hard to see how to prove or disprove this. However, numerical computations of Hejhal in the 1980's decisively disproved any simple connection.)

Keep in mind that, in terms of the geometric objects on which the functions live, the discussion below only refers to a *single space* in a projective family of spaces $\Gamma\backslash\mathfrak{H}$. Thus, for the time being, we are accidentally

[1] Any broadly useful sense of *automorphic form* must be flexible. For the moment, a function f on a space X on which a group Γ acts is *automorphic* (with respect to Γ) if it is Γ-invariant, that is, if $f\gamma x) = f(x)$ for all $\gamma \in \Gamma$ and $x \in X$.

neglecting the actions of the groups $SL_2(\mathbb{Q}_p)$ on the larger projective family. This temporary narrowing of focus is sensible in light of the increased complication of the individual spaces $\Gamma\backslash\mathfrak{H}$, in contrast to circles and lines.

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1. *Elementary eigenfunctions for* $\Delta^{\mathfrak{H}}$

Before coming to the fairly sophisticated issue of $\Delta^{\mathfrak{H}}$ eigenfunctions on quotients $\Gamma\backslash\mathfrak{H}$, we look at more elementary eigenfunctions.

The subgroup

$$N_{\mathbb{R}} \;=\; \{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \;:\; t \in \mathbb{R}\}$$

of $G = SL_2(\mathbb{R})$ acts simply on points $z = x + iy \in \mathfrak{H}$, by translations of the real part:

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} z \;=\; z + t \;=\; (x+t) + iy$$

Asking for $N_{\mathbb{R}}$-invariant $\Delta^{\mathfrak{H}}$-eigenfunctions is to ask for eigenfunctions which depend only upon the imaginary part of z. That is, for f depending upon $y = \mathrm{Im}\, z$ the condition is

$$\Delta^{\mathfrak{H}} f(y) \;=\; y^2 \frac{\partial^2}{\partial y^2}\, f(y) \;=\; \lambda\, f(y)$$

or

$$y^2\, f'' - \lambda\, f \;=\; 0$$

This is an Euler-type equation, meaning that the k^{th} derivative $f^{(k)}$ is multiplied by y^k. Such equations are readily solved in elementary terms, at least for typical values of λ, because the differential operator $y\partial y$ has some obvious eigenvectors y^s with eigenvalues s. Of course, $y^2(\partial y)^2$ is not quite the square of the operator $y\partial y$, so $y^2(\partial y)^2$ acts on y^s by $s(s-1)$ rather than s^2, and so on.

Therefore, a general Euler-type equation

$$y^n\, f^{(n)} + c_{n-1} y^{n-1}\, f^{(n-1)} + \ldots + c_1 y\, f' + c_0\, f \;=\; 0$$

is suspected to have solutions of the form y^s for various $s \in \mathbb{C}$. The complex s satisfying this are found by substituting y^s for f in the equation, obtaining

$$y^n\, s(s-1)(s-2)\ldots(s-n+1) y^{s-n} + c_{n-1} y^{n-1}\, s(s-1)(s-2)\ldots(s-n+2) y^{s-n+1} + \ldots + c_1 y\, s y^{s-1} + c_0\, y^s \;=\; 0$$

which, upon dividing through by y^s, gives a polynomial equation for s,

$$s(s-1)(s-2)\ldots(s-n+1) + c_{n-1} s(s-1)(s-2)\ldots(s-n+2) + \ldots + c_1 s + c_0 \;=\; 0$$

In the case at hand, the equation is

$$s(s-1) - \lambda \;=\; 0$$

[1.0.1] Remark: Various L^2 conditions on the functions involved would assure that $\lambda \le 0$ for eigenfunctions of $\Delta^{\mathfrak{H}}$. This condition implies that s is either of the form $\frac{1}{2} + it$ with t real, or on the real interval $[0,1]$. The resemblance of the picture of this set (the union of the two possibilities) to supposed possibilities for the Riemann-Euler zeta function (and others) is provocative, but turns out to be due to less profound causes.

For $0 \ne \lambda \in \mathbb{C}$, the equation has *distinct* roots [2] s and $1-s$. Thus, for $0 \ne \lambda$, it is easy to show that the general solution of the equation is of the form $ay^s + by^{1-s}$ for $a, b \in \mathbb{C}$.

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[2] We do *not* express the parameter s in terms of the eigenvalue λ by the quadratic formula.

In the degenerate case of multiple roots $s = 1 - s = \frac{1}{2}$, that is, when $\lambda = 0$, in addition to $y^{1/2}$ there is the second solution $y^{1/2} \log y$, which is expressible as a limit expression in terms of the two generic solutions, as

$$y^{1/2} \log y \;=\; \lim_{s \to 1/2} \frac{y^s - y^{1-s}}{s - (1-s)}$$

Since we know that $\Delta^{\mathfrak{H}}$ is $SL_2(\mathbb{R})$-invariant, for any $g \in SL_2(\mathbb{R})$ the translate of y^s (that is, $(\mathrm{Im}\, z)^s)$) by g is also a $\Delta^{\mathfrak{H}}$ eigenfunction with the same eigenvalue. It is straightforward to check that

$$\mathrm{Im}\,(\begin{bmatrix} a & b \\ c & d \end{bmatrix} z) \;=\; \frac{\mathrm{Im}\, z}{|cz+d|^2} \qquad\qquad (\text{for } z \in \mathfrak{H} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R}))$$

Therefore, any function

$$z \to \left(\frac{y}{|cz+d|^2}\right)^s \qquad\qquad (c, d \text{ in } \mathbb{R})$$

is an eigenvector for $\Delta^{\mathfrak{H}}$, with eigenvalue $s(s-1)$.

[1.0.2] Remark: One should observe that Euler equations also have *non-classical* solutions on the real line, as opposed to just the positive real axis. For example,

$$x \frac{\partial}{\partial x} u + u \;=\; 0$$

has the Dirac delta distribution (at 0) as a solution, in addition to y^{-1} and y^0. However, when restricted to the *positive* real axis, the change of variables $y = e^x$ converts Euler equations to linear constant coefficient equations, whose solutions we understand very well (exponentials and variations thereupon). That is, the anomalous solutions are supported on the anomalous *orbit* $\{0\}$ of the group of positive reals acting by dilation on $\mathbb{R}$.

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2. *Less elementary eigenfunctions, and Bessel functions*

The $SL_2(\mathbb{R})$-invariance of $\Delta^{\mathfrak{h}}$ assures without computation that

$$\Delta^{\mathfrak{H}} \frac{y^s}{(x^2+y^2)^s} \;=\; \Delta^{\mathfrak{H}} \left(\mathrm{Im}\,(-1/z)\right)^s \;=\; s(s-1) \cdot \left(\mathrm{Im}\,(-1/z)\right)^s$$

This inverted version of y^s is no longer invariant under the (real) translation group $N_{\mathbb{R}}$, but we can use Fourier transforms to express it as a superposition of $\Delta^{\mathfrak{h}}$ eigenfunctions (all with the same eigenvalue) associated with varying exponentials.

More precisely, but naively, we could contemplate the Fourier transform of $y^s/(x^2+y^2)^s$ in x,

$$\varphi_\xi(y) \;=\; \int_{-\infty}^{\infty} e^{-2\pi i \xi x} \, \frac{y^s}{(x^2+y^2)^s} \, dx$$

[2.0.1] Claim: We have the eigenfunction property

$$\Delta^{\mathfrak{H}} \left(e^{2\pi i \xi x} \, \varphi_\xi(y)\right) \;=\; s(s-1) e^{2\pi i \xi x} \, \varphi_\xi(y)$$

[2.0.2] Remark: It is all too easy to slip into imagining that the Fourier transform $\varphi_\xi(y)$ *without the exponential* is the eigenfunction. It is not.

Proof: We can write the Fourier transform *with the exponential term* $e^{2\pi i i x}$ as a superposition of translates of $f(z) = y^s/(x^2+y^2)^s$, as follows. For $g \in G$, let $g \cdot f$ be the function

$$(g \cdot f)(z) \;=\; f(gz)$$

Then, whenever the integral is reasonably behaved, for any (integrable) function η on N

$$\int_{N_{\mathbb{R}}} \eta(n)\; n \cdot f\, dn$$

should be an $s(s-1)$-eigenfunction for $\Delta^{\mathfrak{H}}$, verified as follows. [3]

$$\begin{array}{rcll}
\Delta^{\mathfrak{H}} \int_{N_{\mathbb{R}}} \eta(n)\; n\cdot f\, dn & = & \int_{N_{\mathbb{R}}} \Delta^{\mathfrak{H}} \eta(n)\; n \cdot f\, dn & (\textit{must} \text{ do this}) \\
& = & \int_{N_{\mathbb{R}}} \eta(n)\; \Delta^{\mathfrak{H}}\; n \cdot f\, dn & (\eta(n) \text{ is a scalar}) \\
& = & \int_{N_{\mathbb{R}}} \eta(n)\, n \cdot (\Delta^{\mathfrak{H}} f)\, dn & (\Delta^{\mathfrak{H}} \text{ commutes with } N) \\
& = & \int_{N_{\mathbb{R}}} \eta(n)\, n \cdot s(s-1)\, f\, dn & \\
& = & s(s-1) \int_{N_{\mathbb{R}}} \eta(n)\, n \cdot f\, dn & (s(s-1) \text{ is scalar})
\end{array}$$

In the situation at hand, for

$$f(z) \;=\; \mathrm{Im}\,(-1/z)^s \;=\; \frac{y^s}{(x^2+y^2)^s}$$

and letting $n = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ we have

$$(n \cdot f)(z) = \frac{y^s}{((t+x)^2+y^2)^s}$$

With $\eta(t) = e^{-2\pi i \xi t}$, the discussion just above assures us that we have a $\Delta^{\mathfrak{H}}$-eigenfunction

$$\int_{\mathbb{R}} e^{-2\pi i \xi t}\, \frac{y^s}{((t+x)^2+y^2)^s}\, dt \;=\; e^{2\pi i \xi x} \cdot \int_{\mathbb{R}} e^{-2\pi i \xi t}\, \frac{y^s}{(t^2+y^2)^s}\, dt$$

by replacing t by $t-x$ in the integral. Thus, the integral (which is a Fourier transform in t) with the exponential term in front is an eigenfunction for $\Delta^{\mathfrak{H}}$. Thus, apart from the interchange of integral and differentiation, the result is easily obtained. ///

The **asymptotic behavior** of the Fourier transform, namely, its rapid decay, is not completely obvious, but is not too difficult to extract from the integral representation, using the fact that the Fourier transform of the Gaussian is essentially itself. The game we play here with integrals is useful in its own right, and is a prototype for many other useful results.

We rearrange the Fourier transform integral to see that (for fixed s) the function $\varphi_\xi(y)$ is really a function of a single parameter, up to elementary factors. Second, the rearranged integral makes sense for *all* $s \in \mathbb{C}$, rather than merely for $\mathrm{Re}\,(s)$ sufficiently positive. Finally, the rearranged integral makes clear that $\varphi_\xi(y)$ is rapidly decreasing as y (or $|\xi|$) becomes large.

[2.0.3] Claim: For $\mathrm{Re}\,(s)$ sufficiently positive, the Fourier transform integral rearranges to

$$\varphi_\xi(y) \;=\; \frac{\pi^s\, |\xi|^{s-\frac{1}{2}}\, y^{\frac{1}{2}}}{\Gamma(s)} \int_0^\infty e^{-\pi(\frac{1}{t}+t)|\xi| y}\, t^{s-\frac{1}{2}}\, \frac{dt}{t} \;=\; |\xi|^{s-1}\, \varphi_1(|\xi| \cdot y)$$

[3] Justification of the interchange of differentiation and integration is not trivial, but we *must* do the interchange, and worry afterward. In fact, under mild hypotheses, the very general Gelfand-Pettis integral decisively addresses this. We delay discussion of Gelfand-Pettis integrals.

[2.0.4] Remark: Note that the integral does not change when $s \to 1-s$, since we can replace t by $1/t$. This is compatible with the fact that both s and $1-s$ produce functions with eigenvalue $s(s-1)$. This **functional equation** is related to similar functional equations of other objects. The following corollary notes that the integral representation gives a **meromorphic continuation** to $\mathbb{C}$, in contrast to the fact that the original integral converges only for $\mathrm{Re}(s) > \frac{1}{2}$.

[2.0.5] Corollary: The rearranged formula for $\varphi_\xi(y)$ makes sense and converges for all complex s (away from non-positive integers, where $\Gamma(s)$ has poles), so gives $\Delta^{\mathfrak{H}}$-eigenvectors for all $s \in \mathbb{C}$.

Proof: (of corollary): An expression $\Delta^{\mathfrak{H}} f_s = s(s-1) \cdot f_s$ with f_s depending holomorphically [4] on s that holds for s in a non-empty open necessarily holds on any connected region where both sides are make sense and are holomorphic (and single-valued). [5] ///

Proof: Use Euler's integral (see appendix)

$$\Gamma(s) \;=\; \int_0^\infty e^{-t}\, t^s \, \frac{dt}{t} \qquad\qquad (\text{with } \mathrm{Re}(s) > 0)$$

and the change-of-variables formula

$$\Gamma(s)\, \alpha^{-s} \;=\; \int_0^\infty e^{-t\alpha}\, t^s \, \frac{dt}{t} \qquad\qquad (\text{for } \alpha > 0 \text{ and } \mathrm{Re}(s) > 0)$$

Applying the latter identity with $\alpha = x^2 + y^2$, for $\mathrm{Re}(s) > 0$

$$\frac{y^s}{(x^2+y^2)^s} \;=\; \frac{y^s}{\Gamma(s)} \int_0^\infty e^{-t(x^2+y^2)}\, t^s \, \frac{dt}{t}$$

Then

$$\varphi_\xi(y) \;=\; \frac{y^s}{\Gamma(s)} \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i \xi x}\, e^{-t(x^2+y^2)}\, t^s \, dx \, \frac{dt}{t}$$

The Gaussian $x \to e^{-tx^2}$ is not $x \to e^{-\pi x^2}$, which would be its own Fourier transform. To rearrange things to this nicer Gaussian, replace x by $x \cdot \frac{\sqrt{\pi}}{\sqrt{t}}$, so

$$\varphi_\xi(y) \;=\; \frac{\sqrt{\pi}\, y^s}{\Gamma(s)} \int_0^\infty \int_{\mathbb{R}} e^{-2\pi i (\sqrt{\pi}\xi/\sqrt{t}) x}\, e^{-\pi x^2}\, e^{-ty^2}\, t^{s-\frac{1}{2}} \, dx \, \frac{dt}{t} \;=\; \frac{\sqrt{\pi}\, y^s}{\Gamma(s)} \int_0^\infty e^{-\pi^2 \xi^2/t}\, e^{-ty^2}\, t^{s-\frac{1}{2}} \, \frac{dt}{t}$$

We can symmetrize the way that $|\xi|$ and y appear inside the integral, to emphasize the fact that there is just *one* essential parameter in this function, not *two*: replace t by $t\pi|\xi|/y$ to obtain

$$\varphi_\xi(y) \;=\; \frac{\pi^s\, |\xi|^{s-\frac{1}{2}}\, y^{\frac{1}{2}}}{\Gamma(s)} \int_0^\infty e^{-\pi(\frac{1}{t}+t)|\xi| y}\, t^{s-\frac{1}{2}} \, \frac{dt}{t} \;=\; |\xi|^{s-1}\, \varphi_1(|\xi| \cdot y)$$

The integral converges for all complex s, since the exponential overwhelms any blow-up of the form t^s at both 0 and $+\infty$. Further, since $t + \frac{1}{t} \geq 2$ for positive real t, the integral is exponentially decaying in $|\xi| y$. ///

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[4] Yes, one might worry for a moment about holomorphic function-valued functions. However, via the Gelfand-Pettis integral we can work out a Cauchy-style theory of holomorphic functions with values in a given quasi-complete locally convex topological vector space, and thereby be assured that essentially all ideas applicable in the scalar-valued situation remain applicable in much broader generality.

[5] This is the *identity principle* from elementary complex analysis.

[2.0.6] Corollary: For $\xi \neq 0$, the function $\varphi_\xi(y)$ is of *rapid decay* as $y \to +\infty$. Specifically, there are finite positive constants A, B (depending on s and ξ) such that

$$|\varphi_\xi(y)| \le A \cdot e^{-By} \qquad \text{(as } y \to +\infty)$$

More precisely,

$$|\varphi_\xi(y)| \le C \cdot |\xi|^{s-\frac{1}{2}} \sqrt{y}\, e^{-\pi|\xi|y}$$

for $\xi \neq 0$, with the constant C depending only upon s. ///

[2.0.7] Corollary: For $\xi \neq 0$, the function $u = \varphi_\xi(y)$ satisfies the second-order differential equation (depending upon the parameter s)

$$u'' + \left(-4\pi^2\xi^2 - \frac{s(s-1)}{y^2}\right) u = 0$$

Proof: This differential equation comes from expanding the eigenfunction assertion

$$s(s-1)\cdot \varphi_\xi\, e^{2\pi i\xi x} \;=\; \Delta^{\mathfrak{H}}\big(\varphi_\xi\, e^{2\pi i \xi x}\big) \;=\; y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\big(\varphi_\xi\, e^{2\pi i\xi x}\big)$$

and removing the exponential function. ///

[2.0.8] Remark: As expected the space of solutions of this differential equation on the positive ray $(0, +\infty)$ is two-dimensional. [6] It is true, but not trivial to prove, that (up to constant multiples) the function $\varphi_n(y)$ is the *only* solution of this differential equation which has the property of *moderate growth* as $y \to +\infty$. Granting this fact for now, the above discussion has shown that among solutions of the differential equation for φ_n, *moderate growth* implies *rapid decay.* That is, there is no middle ground in terms of growth conditions.

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3. *Eisenstein series*

The most immediate Eisenstein series that we can write down without preparation is

$$\widetilde{E}_s(z) \;=\; \tfrac{1}{2}\sum_{c,d} \frac{y^s}{|cz+d|^{2s}} \qquad (c, d \text{ integers, not both } 0,\ z \in \mathfrak{H})$$

It's not too hard to check that this converges absolutely (and uniformly for z in compacts) for $\mathrm{Re}(s) > 1$.

To best understand why $\widetilde{E}_s$ is a $\Delta^{\mathfrak{H}}$-eigenfunction, and to appreciate better the structural aspects of it, we rewrite it so as to express it as a *sum of translates* of the simple eigenfunction $(\mathrm{Im}\, z)^s$.

First, we can pull out the greatest common divisor of each pair c, d, and rearrange the sum to sum over the *gcd*'s first. Thus,

$$\sum_{c,d \text{ not both } 0} \frac{y^s}{|cz+d|^{2s}} \;=\; \sum_{\delta=1}^{\infty} \frac{1}{\delta^{2s}} \sum_{c,d \text{ coprime}} \frac{y^s}{|cz+d|^{2s}} \;=\; \zeta(2s) \sum_{c,d \text{ coprime}} \frac{y^s}{|cz+d|^{2s}}$$

where as usual ζ is the Euler-Riemann zeta function

$$\zeta(s) = \sum_{n \ge 1} \frac{1}{n^s}$$

[6] The differential equation satisfied by φ_ξ is close to being a version of *Bessel's equation.*

Further, changing both c and d by the units $\pm 1 \in \mathbb{Z}^\times$ has no impact on any inner summand. Since there are two units ± 1, we finally have

$$\sum_{c,d \text{ not both } 0} \frac{y^s}{|cz+d|^{2s}} = 2\zeta(2s) \sum_{(c,d) \bmod \pm 1 \text{ coprime}} \frac{y^s}{|cz+d|^{2s}}$$

In anticipation of the following claim, define the *genuine* Eisenstein series

$$E_s(z) \;=\; \sum_{(c,d) \bmod \pm 1 \text{ coprime}} \frac{y^s}{|cz+d|^{2s}}$$

[3.0.1] Claim: The set of pairs (c,d) of coprime integers modulo ± 1 is in natural bijection with lower halves of elements $\begin{bmatrix} * & * \\ c & d \end{bmatrix}$ of $SL_2(\mathbb{Z})$ left-modulo

$$P_{\mathbb{Z}} \;=\; \{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \in SL_2(\mathbb{Z})\}$$

by the correspondence

$$\{\pm 1\} \cdot (c,d) \longleftrightarrow P_{\mathbb{Z}} \begin{bmatrix} * & * \\ c & d \end{bmatrix}$$

Proof: On one hand, for integers c,d not both 0 there are integers a,b such that $ad - bc = \gcd(c,d)$. In particular, for relatively prime c,d, there are a,b with $ad-bc=1$. That is, relatively prime c,d *can* appear as the lower row in a 2-by-2 integer matrix with determinant 1. On the other hand, suppose two matrices have the same bottom row

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \quad \text{in} \quad SL_2(\mathbb{Z})$$

Then

$$\begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} a' & b' \\ c & d \end{bmatrix} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} * & * \\ cd-dc & * \end{bmatrix} \in P_{\mathbb{Z}}$$

That is, the two differ by an upper triangular matrix. ///

Again, this more structured description of the summation encourages us to *redefine* the Eisenstein series, removing the factor of $2\zeta(2s)$ from the original lattice-oriented definition:

$$E_s(z) \;=\; \sum_{\gamma \in P_{\mathbb{Z}} \backslash SL_2(\mathbb{Z})} (\operatorname{Im} \gamma z)^s$$

[3.0.2] Corollary: The sum of terms $y^s/|cz+d|^{2s}$ over relatively prime c,d modulo ± 1 is a sum of *translates* of y^s, namely

$$\sum_{(c,d) \bmod \pm 1 \text{ coprime}} \frac{y^s}{|cz+d|^{2s}} \;=\; \sum_{\gamma \in P_{\mathbb{Z}} \backslash SL_2(\mathbb{Z})} (\operatorname{Im} \gamma z)^s$$

and is therefore a $\Delta^{\mathfrak{H}}$ eigenfunction with eigenvalue $s(s-1)$, at least with $\operatorname{Re}(s) > 1$ for convergence. ///

[3.0.3] Remark: It is important to know that E_s has a *meromorphic continuation* to $s \in \mathbb{C}$. Meromorphic continuation is not a trivial matter. There are several different proofs that apply here, and will be explained later.

4. *Fourier expansions, cuspforms*

Now let f be a $\Delta^{\mathfrak{H}}$-eigenfunction (so is implicitly twice continuously differentiable). Suppose that f is **of moderate growth** as $y \to +\infty$, in the sense that there is a constant C and an exponent N such that (with an arbitrary but fixed lower bound $y_o > 0$ for the imaginary part y)

$$|f(x+iy)| \le C \cdot y^N \qquad \text{(for all } x \in \mathbb{R}, \text{ for } y \ge y_o)$$

[4.0.1] Remark: At other moments we are certainly also interested in an L^2 condition, but the moderate growth condition turns out to be most convenient as a starting place.

Since f is invariant under

$$N_{\mathbb{Z}} \;=\; \{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \;:\; n \in \mathbb{Z}\}$$

the function $f(x+iy)$ has a Fourier expansion [7] in x with coefficients which are functions of y

$$f(x+iy) \;=\; \sum_{n\in\mathbb{Z}} f_n(y)\, e^{2\pi i n x}$$

The integral expression for each Fourier coefficient $f_n(y)$ shows that each $f_n(y)$ is also of moderate growth:

$$|f_n(y)| \;=\; \Big|\int_0^1 e^{-2\pi i n x}\, f(x+iy)\, dx\Big| \le \int_0^1 |f(x+iy)|\, dx \le \int_0^1 C\, y^N\, dx \;=\; C\, y^N$$

Writing an eigenvalue as $s(s-1)$, and presuming that we can differentiate the Fourier series termwise, [8] the eigenfunction condition $\Delta^{\mathfrak{H}} f = s(s-1)\cdot f$ becomes

$$\sum_n \big((2\pi i n)^2 y^2\, f_n(y)\, e^{2\pi i n x} + y^2 f_n''(y)\, e^{2\pi i n x}\big) \;=\; s(s-1) \sum_n f_n(y)\, e^{2\pi i n x}$$

That is, for each integer index n

$$(2\pi i n)^2 y^2\, f_n + y^2 f_n'' \;=\; s(s-1)\, f_n$$

which rearranges to the equation we saw earlier, namely

$$f_n'' + (-4\pi^2 n^2 - \frac{s(s-1)}{y^2})\, f_n \;=\; 0$$

As we saw earlier, for $n = 0$ there are *elementary* solutions [9] y^s and y^{1-s}, which are *both* of moderate growth.

For $n \neq 0$ we already have *one* (non-elementary) solution to this equation, the function φ_n discussed earlier, expressed uniformly for $n \neq 0$ as

$$\varphi_n(y) \;=\; \frac{\pi^s\, |n|^{s-\frac{1}{2}}\, y^{\frac{1}{2}}}{\Gamma(s)} \int_0^\infty e^{-\pi(\frac{1}{t}+t)|n|y}\, t^{s-\frac{1}{2}}\, \frac{dt}{t} \;=\; |n|^{s-1}\, \varphi_1(|n|y)$$

[7] Since f is at least C^2, this series converges (absolutely) pointwise in x.

[8] We saw earlier that termwise differentiation of Fourier series is legitimate at least for distributions on the circle, and also as a map $\mathrm{C}^k \to \mathrm{C}^{k-1}$ for suitable k.

[9] And when $s = 1/2$ there is also $y^{1/2}\,\log y$, which does arise as the limit of $(y^s - y^{1-s})/(s-\frac{1}{2})$ as s goes to $\frac{1}{2}$.

This is not only of *moderate growth*, but of *rapid decay*.

Granting for the moment that (apart from scalar multiples of φ_n) no other solution of the differential equation is of moderate growth, we can rewrite the Fourier expansion with *numerical* Fourier coefficients a, b, c_n as

$$f(x+iy) \;=\; ay^s + by^{1-s} + \sum_{n\neq 0} c_n\,\varphi_1(|n|y)\,e^{2\pi i n x}$$

When the constant (in x) coefficients a and b are 0, the $\Delta^{\mathfrak{H}}$-eigenfunction f is said to be a **cuspform**. [10]

[4.0.2] Theorem: A moderate-growth cuspform f is (in fact) *of rapid decay* as $y \to +\infty$.

Proof: From the Fourier expansion

$$f(x+iy) \;=\; ay^s + by^{s-1} + \sum_{n\neq 0} c_n\,\varphi_1(|n|y)\,e^{2\pi i n x}$$

and an easy estimate

$$\varphi_1(y) \le C\cdot \sqrt{y}\,e^{-\pi y}$$

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[4.0.3] Remark: The previous proof makes needless use of *details* about the Bessel-like functions $\varphi_1(|n|y)$. We will give a smoother proof later.

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5. *Appendix: the Gamma function*

The Euler integral

$$\Gamma(s) \;=\; \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t}$$

for the gamma function $\Gamma(s)$ makes sense for complex s with $\mathrm{Re}(s) > 0$. Integrating by parts gives the basic functional equatiopn of $\Gamma(s)$, namely

$$\Gamma(s) \;=\; \int_0^{+\infty} e^{-t}\,t^s\,\frac{dt}{t} \;=\; \frac{\Gamma(s+1)}{s}$$

valid at least for $\mathrm{Re}(s) > 0$. Iterating this, we obtain expressions

$$\Gamma(s) \;=\; \frac{\Gamma(s+n+1)}{s(s+1)(s+2)\ldots(s+n-1)(s+n)}$$

The right-hand side makes sense for $\mathrm{Re}(s) > -(n+1)$, apart from the obvious poles at $0, -1, -2, \ldots, -n$, giving a meromorphic extension of Γ to $\mathrm{Re}(s) > -(n+1)$. This trick works for all n, so Γ is meromorphic on $\mathbb{C}$, with readily determinable residues.

The previous identity with $s = 1$ gives

$$1 \;=\; \int_0^\infty e^{-t}\,dt \;=\; \Gamma(1) \;=\; \frac{\Gamma(n+1)}{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\,n}$$

[10] The terminology *cusp* refers to neighborhoods of the infinitely-distant missing point on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, which is often pictorially represented as being *pointy*, whence the name. More literally, it refers the behavior as $y \to +\infty$, but the colorful quasi-geometric terminology is both traditional and memorable.

which shows

$$\Gamma(n+1) \;=\; n! \qquad\qquad (n \text{ a non-negative integer})$$

We also need the identity obtained by replacing t by Ct for a positive constant C,

$$\Gamma(s)\, C^{-s} \;=\; \int_0^{+\infty} e^{-Ct}\, t^s\, \frac{dt}{t}$$

This is valid first for $\mathrm{Re}(s) > 0$. Since both sides are holomorphic functions (off non-positive integers) and agree on a non-empty open set, they are equal everywhere. [11]

This integral yields a useful identity by replacing y by ny for positive n, namely

$$\Gamma(s)\, \frac{1}{n^s} \;=\; \int_0^{\infty} e^{-ny}\, y^s\, \frac{dy}{y}$$

Applying this to a sum

$$f(y) \;=\; \sum_{n=1}^{\infty} c_n\, e^{-2\pi n y}$$

and (assuming convergence)

$$(2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{c_n}{n^s} \;=\; \int_0^{\infty} e^{-2\pi n y}\, y^s\, \frac{dy}{y} \;=\; \int_0^{\infty} f(y)\, y^s\, \frac{dy}{y}$$

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[11] The *identity principle* from complex analysis asserts that two holomorphic functions on a connected open which agree on a non-empty set possessing a limit point must agree on the entire open set.